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KILLING SUPERALGEBRAS FOR LORENTZIAN SIX-MANIFOLDS

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ABSTRACT. We calculate the Spencer cohomology of the $(1,0)$ Poincaré superalgebras in six dimensions: with and without R-symmetry. As the cases of four and eleven dimensions taught us, we may read off from this calculation a Killing spinor equation which allows the determination of which geometries admit rigidly supersymmetric theories in this dimension. We prove that the resulting Killing spinors generate a Lie superalgebra and determine the geometries admitting the maximal number of such Killing spinors. They are divided in two branches. One branch consists of the lorentzian Lie groups with bi-invariant metrics and, as a special case, it includes the lorentzian Lie groups with a self-dual Cartan three-form which define the maximally supersymmetric backgrounds of $(1,0)$ Poincaré supergravity in six dimensions. The notion of Killing spinor on the other branch does not depend on the choice of a three-form but rather on a one-form valued in the R-symmetry algebra. In this case, we obtain three different (up to local isometry) maximally supersymmetric backgrounds, which are distinguished by the causal type of the one-form.

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1. Introduction

There has been considerable interest over recent years in the systematic exploration of curved backgrounds that support some amount of rigid (conformal) supersymmetry. The primary motivation being that quantum field theories on such backgrounds are often amenable to the powerful techniques of supersymmetric localisation, typically revealing interesting new insights and exact results [1–9].

By far the most successful strategy in this direction was initiated by Festuccia and Seiberg [10], originally for rigidly supersymmetric backgrounds in four dimensions but subsequently generalised [11–18] to other dimensions in both euclidean and lorentzian signatures. Their method takes advantage of the existence of some locally supersymmetric supergravity theory coupled to one or more field theory supermultiplets. In any such theory, it is possible to take a certain rigid limit in which the Planck mass tends to infinity and the degrees of freedom from the gravity supermultiplet are effectively frozen out. What remains after taking this limit is a rigidly supersymmetric field theory on a bosonic supersymmetric background of the original supergravity theory. The Killing spinor equations which characterise this supersymmetric background are simply read off from the supersymmetry variation of the gravitino in the rigid limit. It is important to emphasise that these supersymmetric backgrounds need not solve the supergravity field equations. For example, in four dimensions, the old minimal off-shell formulation of Poincaré supergravity contains auxiliary fields which are all set to zero by the field equations. However, many interesting rigidly supersymmetric backgrounds of this theory are not solutions because they are supported by one or more non-zero auxiliary fields [10, 19].

The precise details of the rigid supersymmetry supported by any bosonic supersymmetric supergravity background are encoded by its Killing superalgebra [20–24]. The Killing superalgebra is a Lie superalgebra whose odd part consists of all the Killing spinors supported by the background and whose even part contains Killing vectors which preserve the background. For supergravity theories with a non-trivial R-symmetry, the even part of the Killing superalgebra may also contain R-symmetries which preserve the background. While the appearance of the Killing superalgebra may seem somewhat peripheral in relation to the rigid limit described above, it is clearly an object of fundamental significance in the description of rigid supersymmetry and in understanding special geometrical properties of the backgrounds which support it.

So much so that, somewhat in the spirit of the Erlangen program, one might prefer to take the classification of Killing superalgebras as the central question, with no prior knowledge of supergravity, and then deduce as a by product all the possible rigidly supersymmetric backgrounds (which may or may not correspond to backgrounds of some known supergravity theory). This is certainly the philosophy we have adopted in some of our previous works, which has led to the classification of Killing superalgebras for maximally supersymmetric lorentzian backgrounds in dimensions eleven [25,26] and four [19]. The key property of Killing superalgebras that permits such a classification is the fact that they are all filtered deformations (in a certain technical sense which we review in §8.5) of some subalgebra of the Poincaré superalgebra, possibly extended by R-symmetries. As one might expect, there is a natural cohomology theory (a generalised version of Spencer cohomology) which governs these filtered deformations at the infinitesimal level, and the essence of the classification is the calculation of a certain Spencer cohomology group in degree two. In dimensions eleven and four [19,25,26], this calculation actually prescribes a Killing spinor equation which is in precise agreement with the Killing spinor equation that characterises bosonic supersymmetric backgrounds of minimal Poincaré supergravity in these respective dimensions (more accurately, in the 'old minimal' off-shell formulation in four dimensions [19]). So, at least in these cases, all the rigidly supersymmetric backgrounds are indeed backgrounds of a known Poincaré supergravity theory.

In this paper, we shall extend these considerations to look at Killing superalgebras for lorentzian backgrounds in six dimensions. There are several reasons that make dimension six especially interesting. Recall that the Lie superalgebra $\mathfrak{osp}(6,2|N)$ is isomorphic to the N-extended conformal superalgebra of

$\mathbb{R}^{5,1}$, and that conformal superalgebras do not exist in higher dimensions (at least, not in the traditional sense of Nahm [27]). Furthermore the $\mathfrak{sp}(N)$ R-symmetry subalgebra of $\mathfrak{osp}(6,2|N)$ is nonabelian, for any $N > 0$. Now let $N = 1$. By omitting the dilatations and special conformal transformations in the even part of $\mathfrak{osp}(6,2|1)$, together with the special conformal supercharges in the odd part, we obtain a Lie superalgebra that we will denote by $\hat{\mathfrak{p}}$. If we also omit the $\mathfrak{sp}(1)$ R-symmetry in $\hat{\mathfrak{p}}$, we recover the ordinary $(1,0)$ Poincaré superalgebra in six dimensions (without R-symmetry) that we will denote by $\mathfrak{p}$.

Our first goal in this paper will be to calculate the relevant Spencer cohomology groups for both $\mathfrak{p}$ and $\hat{\mathfrak{p}}$, see Theorems 10 and 11. In marked contrast with the situation in dimensions eleven and four, where the inclusion of R-symmetry is immaterial, here in dimension six we will find that the relevant Spencer cohomology groups for $\mathfrak{p}$ and $\hat{\mathfrak{p}}$ are different. In both cases, we then go on to use the explicit expression for a particular component of the Spencer cocycle representative to prescribe an appropriate Killing spinor equation. For the case of ungauged R-symmetry, this Killing spinor equation is given by Definition 12 in Section 6 while, for the case of gauged R-symmetry, the Killing spinor equation is defined by (157) in Section 7. In both cases, on a six-dimensional spin manifold M equipped with a lorentzian metric g, we find that the extra background data needed to define this Killing spinor equation consists of a three-form H and an $\mathfrak{sp}(1)$-valued one-form φ. (For the case of gauged R-symmetry, one must also specify a flat $\mathfrak{sp}(1)$ connection C.) The important distinction is that H must be self-dual for $\mathfrak{p}$ whereas, for $\hat{\mathfrak{p}}$, its anti-self-dual component H^- need not be zero. It is important to stress that the Killing spinor equation we deduce from Spencer cohomology agrees with the Killing spinor equation for bosonic supersymmetric backgrounds of $(1,0)$ Poincaré supergravity in six dimensions *only* when H is self-dual and $\varphi = 0$. (For the case of gauged R-symmetry, one can remove the flat connection C by an appropriate choice of gauge.) It is an intriguing question as to whether our more general Killing spinor equation can be recovered from supergravity, perhaps via superconformal compensators.

We then proceed to the construction of Killing superalgebras based on these Killing spinors. The geometric content of the cocycle conditions for Spencer cohomology is that the Dirac current of a Killing spinor (derived from Spencer cohomology) is a Killing vector and that the Lie derivative along the Dirac current annihilates the Killing spinor itself. In order to prove that Killing spinors generate a Lie superalgebra, the only additional requirement is that the Lie derivative along the Dirac current of any Killing spinor preserves the space of Killing spinors. This is guaranteed if the connection $\mathscr{D}$ defining the notion of a Killing spinor is invariant along the flow generated by the Dirac current of any Killing spinor.

An equivalent condition for the invariance of $\mathscr{D}$ is the invariance of the other geometric data defining $\mathscr{D}$: the three-form H and $\mathfrak{sp}(1)$-valued one-form φ. For $\mathfrak{p}$, we establish the existence of a Killing superalgebra provided H is closed and φ is coclosed, see Theorem 20. For $\hat{\mathfrak{p}}$, if $\varphi = 0$, we find that a Killing superalgebra exists provided H is closed and H^- is parallel with respect to the metric connection with skew-symmetric torsion given by H^+, see Theorem 24.

Finally, we present in Theorem 27 the classification (up to local isometry) of all backgrounds which admit the maximal number of Killing spinors. In addition to Minkowski space $\mathbb{R}^{5,1}$, we find that there are two distinct branches of maximally supersymmetric backgrounds. All backgrounds on the first branch are conformally flat and have $H = 0$ with $\varphi = \alpha \otimes R$, where α is a non-zero parallel one-form and R is a non-zero element of $\mathfrak{sp}(1)$. Up to local isometry, there are three different backgrounds on this branch which depend only on the causal type of α:

- $\mathrm{AdS}_5 \times \mathbb{R}$, if α is spacelike;
- $\mathbb{R} \times S^5$, if α is timelike;
- the symmetric plane wave with metric g_- in (242), if α is null.

All backgrounds on the second branch have $\varphi = 0$ and a non-zero H which is identified with the parallel Cartan three-form of a six-dimensional Lie group M with bi-invariant lorentzian metric g. Up to local isometry, the list of different backgrounds on this branch is as follows:

- $\mathrm{AdS}_3 \times S^3$;

- $\mathrm{AdS}_3 \times \mathbb{R}^3$;
- $\mathbb{R}^{2,1} \times \mathrm{S}^3$;
- the symmetric plane wave with metric g_- in (242).

In the first three cases, H can be any linear combination (with non-zero coefficients) of the volume forms on the respective AdS_3 and S^3 factors. If H is non-zero and self-dual, only the first and fourth cases above are viable, and we recover precisely the classification [28,29] of maximally supersymmetric backgrounds of $(1,0)$ Poincaré supergravity in six dimensions.

In conclusion, we verify that all of these maximally supersymmetric backgrounds do indeed admit a Killing superalgebra and that different backgrounds have different associated Killing superalgebras, in the sense of filtered deformations.

This paper is organised as follows. In Section 2 we introduce our six-dimensional spinor conventions, set the notation and prove a number of algebraic results that we will use in the rest of the paper. In Section 3 we introduce the Spencer cohomology complexes associated to the $(1,0)$ Poincaré superalgebra $\mathfrak{p}$ and its extension $\hat{\mathfrak{p}}$ by the R-symmetry. The relevant cohomology groups are computed in Section 4 for $\mathfrak{p}$ and in Section 5 for $\hat{\mathfrak{p}}$. From these calculations we extract the Killing spinor equations and in Section 6 we show that, subject to some additional conditions on the geometric data given by the Spencer cohomology, these Killing spinors generate a Lie superalgebra. This is revisited in Section 7 in a slightly different formalism, paying particularly close attention to the case of gauged R-symmetry. Finally, in Section 8 we determine the geometries admitting the maximal number of Killing spinors. These are then the candidate six-dimensional lorentzian manifolds on which to construct rigidly supersymmetric theories with eight real supercharges.

2. Conventions

Let (V,η) be a six-dimensional ("mostly plus") lorentzian vector space. We may choose a η-orthonormal basis $(\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_5)$ for V relative to which $\eta(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu) = \eta_{\mu\nu} = \mathrm{diag}(-1,+1,\dots,+1)$.

We will let $\flat : V \to V^*$ and $\sharp : V^* \to V$ denote the musical isomorphisms:

$$v^\flat(w) = \eta(v,w) \qquad \text{and} \qquad \xi(v) = \eta(\xi^\sharp, v). \tag{1}$$

It follows, as usual, that $\flat$ and $\sharp$ are mutual inverses. We let $\mathfrak{so}(V)$ be the Lie algebra of η-skew-symmetric endomorphisms of V:

$$\mathfrak{so}(V) = \{L : V \to V \mid \eta(Lv, w) = -\eta(v, Lw) \quad \forall v, w \in V\}. \tag{2}$$

There is a vector space (in fact, an $\mathfrak{so}(V)$-module) isomorphism $\mathfrak{so}(V) \cong \Lambda^2 V$. If $L \in \mathfrak{so}(V)$, we define $\omega_L \in \Lambda^2 V$ by

$$Lv = -\iota_{v^\flat}\omega_L. \tag{3}$$

Conversely, if $\omega \in \Lambda^2 V$, we define $L_\omega \in \mathfrak{so}(V)$ by the same relationship: namely,

$$L_\omega v = -\iota_{v^\flat}\omega. \tag{4}$$

It then follows that these two maps are mutual inverses: $L_{\omega_L} = L$ and $\omega_{L_\omega} = \omega$. Relative to the basis $(\boldsymbol{e}_\mu)$ for V, we find that

$$\omega_L = \tfrac{1}{2} L^{\mu\nu} \boldsymbol{e}_\mu \wedge \boldsymbol{e}_\nu \qquad \text{where} \qquad L\boldsymbol{e}_\mu = \boldsymbol{e}_\nu L^\nu{}_\mu. \tag{5}$$

We define the Clifford algebra $C\ell(V)$ by the Clifford relations

$$v \cdot v = \eta(v,v)\mathbb{1}. \tag{6}$$

As a real unital associative algebra, $C\ell(V) \cong \mathbb{H}(4)$, whereas we have an isomorphism of Lie groups $\mathrm{Spin}(V) \cong \mathrm{SL}(2,\mathbb{H})$. There is, up to isomorphism, a unique irreducible Clifford module Σ which is quaternionic and of dimension 4. We prefer to think of Σ as an 8-dimensional complex vector space with an invariant quaternionic structure. As a representation of $\mathrm{Spin}(V)$ it breaks up as $\Sigma = \Sigma_+ \oplus \Sigma_-$, where $\Sigma_\pm$ are irreducible representations: either quaternionic of dimension 2 or, equivalently, complex

4-dimensional with an invariant quaternionic structure. Let Δ denote the fundamental representation of $\mathrm{Sp}(1)$: it can be thought of as a complex 2-dimensional representation with an invariant quaternionic structure. The tensor product $\Sigma_+ \otimes_{\mathbb{C}} \Delta$ is the complexification of a real representation of $\mathfrak{so}(V)$ we call S. In other words, $S \otimes \mathbb{C} = \Sigma_+ \otimes_{\mathbb{C}} \Delta$. In practice we prefer to work with $S \otimes \mathbb{C}$; although we will not mention this explicitly.

There is a dual pairing $\langle -, - \rangle$ between Σ_+ and Σ_- relative to which,

$$\langle v \cdot s_+, s_- \rangle = -\langle s_+, v \cdot s_- \rangle \tag{7}$$

for all $v \in V$ and $s_\pm \in \Sigma_\pm$. We may extend it to a *symmetric* inner product on $\Sigma = \Sigma_+ \oplus \Sigma_-$, also denoted $\langle -, - \rangle$, in such a way that $\Sigma_\pm$ are (maximally) isotropic subspaces. We will use the notation $\overline{s} = \langle s, - \rangle$, so that $\overline{s}_1 s_2 = \langle s_1, s_2 \rangle$.

If we let $\boldsymbol{\epsilon}_A$, $A = 1, 2$, denote a basis for Δ, any $s \in S$ can be written as $s = s^A \boldsymbol{\epsilon}_A$; we will often just work with the components $s^A \in \Sigma_+$. On Δ we have an invariant symplectic structure ϵ, normalised to $\epsilon_{12} = \epsilon^{12} = 1$. In §6, we will also make use of the skew-symmetric bilinear form $(-, -)$ on $\Sigma \otimes_{\mathbb{C}} \Delta$ given by the tensor product of $\langle -, - \rangle$ and ϵ.

We use the Northeast convention to raise and lower indices with ϵ:

$$u_A = \epsilon_{AB} u^B \qquad \text{and} \qquad u^A = u_B \epsilon^{BA}, \tag{8}$$

from where it follows that $\epsilon_{AB}\epsilon^{AC} = \delta_B^C$. Since Δ is 2-dimensional, $\Lambda^2\Delta^*$ is one-dimensional and spanned by ϵ. Any bivector $B \in \otimes^2\Delta \cong \otimes^2\Delta^*$ can be decomposed into $B_{AB} = (B_0)_{AB} + \epsilon_{AB} b$, where $(B_0)_{AB} = (B_0)_{BA}$ and $b = \frac{1}{2} B_{AB}\epsilon^{AB}$. The notation suggests that if we think of the bivector as an endomorphism of Δ, then $B^A{}_B = (B_0)^A{}_B + b\delta_B^A$ and B_0 is traceless.

The basis element $\boldsymbol{e}_\mu$ for V is sent to the endomorphism of Σ denoted Γ_μ, where

$$\Gamma_\mu \Gamma_\nu = \Gamma_{\mu\nu} + \eta_{\mu\nu}. \tag{9}$$

We define the volume in the Clifford algebra by $\Gamma_7 := \Gamma_{012345}$ and the projection operators $P_\pm = \frac{1}{2}(\mathbb{1} \pm \Gamma_7)$. Then $\Gamma_7 s^A = s^A$ and hence $P_+ s^A = s^A$.

We can make two kinds of bilinears from $s \in S$: the *Dirac current* $\kappa = \kappa(s, s) \in V$ with components

$$\kappa^\mu = \epsilon_{AB} \overline{s}^A \Gamma^\mu s^B \iff \overline{s}^A \Gamma^\mu s^B = \tfrac{1}{2}\epsilon^{AB}\kappa^\mu, \tag{10}$$

and a family ω of 3-forms given by

$$\omega_{\mu\nu\rho}{}^{AB} = \overline{s}^A \Gamma_{\mu\nu\rho} s^B. \tag{11}$$

Lemma 1. $\omega \in \Lambda^3_+ V \otimes \odot^2 \Delta$.

Proof. Only the self-duality needs proof. We calculate

$$\begin{aligned} \omega_{\mu\nu\rho}{}^{AB} &= \overline{s}^A \Gamma_{\mu\nu\rho} s^B \\ &= \overline{s}^A \Gamma_{\mu\nu\rho} \Gamma_7 s^B \\ &= \tfrac{1}{6} \varepsilon_{\mu\nu\rho\lambda\sigma\tau} \overline{s}^A \Gamma^{\lambda\sigma\tau} s^B \\ &= \tfrac{1}{6} \varepsilon_{\mu\nu\rho\lambda\sigma\tau} \omega^{\lambda\sigma\tau\, AB} \\ &= (\star\omega)_{\mu\nu\rho}{}^{AB}. \end{aligned} \tag{12}$$

□

A very useful identity is the *Fierz identity*, which says

$$s^A \overline{s}^B = -\tfrac{1}{8}\left(\epsilon^{AB}\kappa + \omega^{AB}\right) P_-, \tag{13}$$

for all $s = s^A \boldsymbol{\epsilon}_A \in S$. An immediate consequence of this identity is that the Dirac current of s Clifford-annihilates s.

Lemma 2. $(\overline{s}^A \Gamma^\mu s^B)\Gamma_\mu s^C = 0$.

Proof. Since $(\overline{s}^A\Gamma^\mu s^B)\Gamma_\mu s^C = \frac{1}{2}\epsilon^{AB}\kappa\cdot s^C$, it is enough to show that $\kappa\cdot s^C = 0$. We calculate

$$\begin{aligned}(\overline{s}^A\Gamma^\mu s^B)\Gamma_\mu s^C &= \Gamma_\mu(s^C\overline{s}^A)\Gamma^\mu s^B\\ &= -\tfrac{1}{8}\Gamma_\mu\left(\epsilon^{CA}\kappa + \omega^{CA}\right)\Gamma^\mu s^B\\ &= \tfrac{1}{2}\epsilon^{CA}\kappa\cdot s^B,\end{aligned} \tag{14}$$

where we have used the useful identities

$$\Gamma^\mu\kappa\Gamma_\mu = -4\kappa \qquad \text{and} \qquad \Gamma^\mu\omega^{AB}\Gamma_\mu = 0. \tag{15}$$

We now contract both sides with ϵ_{AB} to arrive at

$$\kappa\cdot s^C = \tfrac{1}{2}\epsilon^{CA}\epsilon_{AB}\kappa\cdot s^B = -\tfrac{1}{2}\kappa\cdot s^C, \tag{16}$$

which shows that $\kappa\cdot s^C = 0$. □

The 3-form ω^{AB} associated to s also Clifford-annihilates s. In fact, more generally, we have the following

Lemma 3. *Let $\Xi\in\Lambda^3_+V$ and $s\in\Sigma_+$. Then $\Xi\cdot s = 0$.*

Proof. The identity

$$\Gamma_{\mu\nu\rho}\Gamma_7 = \tfrac{1}{3!}\varepsilon_{\mu\nu\rho\lambda\sigma\tau}\Gamma^{\lambda\sigma\tau}, \tag{17}$$

implies that if $\Xi\in\Lambda^3V$, then $\Xi\Gamma_7 = -\star\Xi$, so that if $s\in\Sigma_+$,

$$\Xi\cdot s = \Xi\Gamma_7\cdot s = -(\star\Xi)\cdot s \implies (\Xi + \star\Xi)\cdot s = 0. \tag{18}$$

If Ξ is self-dual, the result follows. □

In the same way one can show that the Clifford product of two self-dual (or antiself-dual) 3-forms vanishes.

Lemma 4. *If $\Xi_1,\Xi_2\in\Lambda^3_\pm V$, then $\Xi_1\Xi_2 = 0$.*

Proof. The identity (17) says that if $\Xi\in\Lambda^3_\pm V$, then $\Xi\Gamma_7 = \mp\Xi$. Now we calculate

$$\Xi_1\Xi_2 = \Xi_1\Gamma_7^2\Xi_2 = (\Xi_1\Gamma_7)(-\Xi_2\Gamma_7) = -\Xi_1\Xi_2. \tag{19}$$

□

The Dirac current and the 3-form satisfy the following properties.

Lemma 5. *Let κ and ω^{AB} be the Dirac current and 3-form associated to a non-zero $s\in S$. Then*

(i) κ is a non-vanishing null vector, and
(ii) the 3-forms ω^{AB}, $A,B = 1,2$, are linearly independent (in particular they do not vanish).

Proof. The Dirac current κ is a null vector from Lemma 2. By the Fierz identity (13) we have

$$s^A\overline{s}^B - s^B\overline{s}^A = -\tfrac{1}{4}\epsilon^{AB}\kappa P_-, \tag{20}$$

and $s = s^A\epsilon_A$ is a decomposable element of the tensor product $\Sigma_+\otimes_{\mathbb{C}}\Delta$ if $\kappa = 0$. This is not possible, since s is a real spinor. The non-vanishing of any ω^{AB} is proved similarly.
Now, making use of the Fierz identity, it is a simple matter to deduce the identity

$$\omega^{AB}_{\mu[\nu}{}^{\tau}\omega^{CD}_{\rho\sigma]\tau} = -\tfrac{1}{3}\kappa_\mu(\epsilon^{A(C}\omega^{D)B}_{\nu\rho\sigma} + \epsilon^{B(C}\omega^{D)A}_{\nu\rho\sigma})\ . \tag{21}$$

Consequently,

$$\omega^{11}_{\mu[\nu}{}^{\tau}\omega^{12}_{\rho\sigma]\tau} = -\tfrac{1}{3}\kappa_\mu\omega^{11}_{\nu\rho\sigma}\ ,\quad \omega^{22}_{\mu[\nu}{}^{\tau}\omega^{12}_{\rho\sigma]\tau} = \tfrac{1}{3}\kappa_\mu\omega^{22}_{\nu\rho\sigma}\ ,\quad \omega^{11}_{\mu[\nu}{}^{\tau}\omega^{22}_{\rho\sigma]\tau} = -\tfrac{2}{3}\kappa_\mu\omega^{12}_{\nu\rho\sigma}\ , \tag{22}$$

are all non-vanishing. Now, assume ω^{11} is a linear combination of ω^{12} and ω^{22} and substitute the former on the LHS of the first equation in (22). Using again (21) with $A = C = 1$ and $B = D = 2$, we get that ω^{11} and ω^{22} are proportional. Plugging this back into the last equation in (22) implies $\omega^{12} = 0$, which is absurd. □

Finally (for now), we have two additional algebraic relations between the Dirac current and the 3-form.

Lemma 6. *Let κ and ω^{AB} be the Dirac current and 3-form associated to $s \in S$. Then*
  (i) $\iota_\kappa \omega^{AB} = 0$*, and*
  (ii) $\kappa^\flat \wedge \omega^{AB} = 0$.

Proof. To prove the first identity, we compute

$$\begin{aligned}\kappa^\rho \omega_{\mu\nu\rho}{}^{AB} &= \kappa^\rho \overline{s}^A \Gamma_{\mu\nu\rho} s^B \\ &= \kappa^\rho \overline{s}^A \left(\Gamma_{\mu\nu}\Gamma_\rho - \eta_{\nu\rho}\Gamma_\mu + \eta_{\mu\rho}\Gamma_\nu\right) s^B \\ &= \overline{s}^A \Gamma_{\mu\nu} \kappa \cdot s^B - \kappa_\nu \overline{s}^A \Gamma_\mu s^B + \kappa_\mu \overline{s}^A \Gamma_\nu s^B.\end{aligned} \tag{23}$$

The first term vanishes because of Lemma 2 and the last two terms precisely cancel each other. The second identity follows from the first due to the self-duality of ω^{AB}:

$$\kappa^\flat \wedge \omega^{AB} = \kappa^\flat \wedge \star\omega^{AB} = \star(\iota_\kappa \omega^{AB}) = 0. \tag{24}$$

□

3. Spencer complexes associated to the $(1,0)$ Poincaré superalgebra

The d=6 $(1,0)$ Poincaré superalgebra is the $\mathbb{Z}$-graded Lie superalgebra

$$\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0 = V \oplus S \oplus \mathfrak{so}(V), \tag{25}$$

with nonzero Lie brackets

$$[L, M] = LM - ML, \qquad [L, v] = Lv, \qquad [L, s] = \tfrac{1}{2}\omega_L \cdot s \qquad \text{and} \qquad [s, s] = \kappa, \tag{26}$$

for all $L, M \in \mathfrak{so}(V)$, $v \in V$ and $s \in S$. We will also consider the extended Poincaré superalgebra $\hat{\mathfrak{p}}$ where in degree zero we have $\hat{\mathfrak{p}}_0 = \mathfrak{so}(V) \oplus \mathfrak{r}$, where $\mathfrak{r} \cong \mathfrak{sp}(1)$ is the R-symmetry.
There are (generalised) Spencer complexes associated to $\mathfrak{p}$ and $\hat{\mathfrak{p}}$ which we now describe. They govern filtered subdeformations of these graded Lie superalgebras.

3.1. Spencer complex of $\mathfrak{p}$.

In the first instance, we will calculate the cohomology of the Spencer complex

$$C^{2,1}(\mathfrak{p}_-, \mathfrak{p}) \xrightarrow{\ \partial\ } C^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \xrightarrow{\ \partial\ } C^{2,3}(\mathfrak{p}_-, \mathfrak{p}), \tag{27}$$

where the spaces of cochains are

$$\begin{aligned} C^{2,1}(\mathfrak{p}_-, \mathfrak{p}) &= \mathrm{Hom}(V, \mathfrak{so}(V)) \\ C^{2,2}(\mathfrak{p}_-, \mathfrak{p}) &= \mathrm{Hom}(\Lambda^2 V, V) \oplus \mathrm{Hom}(V \otimes S, S) \oplus \mathrm{Hom}(\odot^2 S, \mathfrak{so}(V)) \\ C^{2,3}(\mathfrak{p}_-, \mathfrak{p}) &= \mathrm{Hom}(V \otimes \odot^2 S, V) \oplus \mathrm{Hom}(\odot^3 S, S) \end{aligned} \tag{28}$$

and the differentials are such that if $\lambda \in C^{2,1}(\mathfrak{p}_-, \mathfrak{p})$, then

$$\partial\lambda(v, w) = \lambda_v w - \lambda_w v, \qquad \partial\lambda(v, s) = \tfrac{1}{2}\omega_{\lambda_v} \cdot s \qquad \text{and} \qquad \partial\lambda(s, s) = -\lambda_{[s,s]}, \tag{29}$$

and if $\psi = \alpha + \beta + \gamma \in C^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ with

$$\alpha : \Lambda^2 V \to V, \qquad \beta : V \otimes S \to S \qquad \text{and} \qquad \gamma : \odot^2 S \to \mathfrak{so}(V), \tag{30}$$

then

$$\partial\psi(v, s, s) = [v, \gamma(s, s)] - 2[s, \beta_v s] - \alpha([s, s], v) \qquad \text{and} \qquad \tfrac{1}{3}\partial\psi(s, s, s) = [s, \gamma(s, s)] - \beta_{[s,s]} s. \tag{31}$$

Lemma 7. *The Spencer differential $\partial : C^{2,1}(\mathfrak{p}_-, \mathfrak{p}) \to C^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ is injective, so that in particular* $H^{2,1}(\mathfrak{p}_-, \mathfrak{p}) = 0$.

Proof. Let $\lambda \in C^{2,1}(\mathfrak{p}_-, \mathfrak{p})$. If $\partial\lambda = 0$, then in particular $\partial\lambda(s, s) = -\lambda_{[s,s]} = 0$. Since $[S, S] = V$, then $\lambda = 0$. □

Moreover it is not just that $\partial : C^{2,1}(\mathfrak{p}_-, \mathfrak{p}) \to C^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ is injective, but that the component $C^{2,1}(\mathfrak{p}_-, \mathfrak{p}) \to \operatorname{Hom}(\Lambda^2 V, V)$ is an isomorphism. This follows because if $\partial\lambda(v, w) = 0$, the tensor $T(u, v, w) := \eta(u, \lambda_v w)$ satisfies $T(u, v, w) = T(u, w, v)$ in addition to $T(u, v, w) = -T(w, v, u)$, so that it is identically zero.
This fact allows us to compute the cohomology by determining the kernel of the Spencer differential on the normalised cochains satisfying $\alpha = 0$. In other words, we have the following

Proposition 8. *There is an isomorphism (of $\mathfrak{so}(V)$-modules)*

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \left\{ \beta + \gamma \in C^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \middle| \partial(\beta + \gamma) = 0 \right\}. \tag{32}$$

In Section 4 we calculate this cohomology.

3.2. **Spencer complex of $\hat{\mathfrak{p}}$.** We will also calculate the cohomology of the Spencer complex

$$C^{2,1}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}}) \xrightarrow{\partial} C^{2,2}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}}) \xrightarrow{\partial} C^{2,3}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}}), \tag{33}$$

where the spaces of cochains are

$$\begin{aligned} C^{2,1}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}}) &= \operatorname{Hom}(V, \mathfrak{so}(V)) \oplus \operatorname{Hom}(V, \mathfrak{r}) \\ C^{2,2}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}}) &= \operatorname{Hom}(\Lambda^2 V, V) \oplus \operatorname{Hom}(V \otimes S, S) \oplus \operatorname{Hom}(\odot^2 S, \mathfrak{so}(V)) \oplus \operatorname{Hom}(\odot^2 S, \mathfrak{r}) \\ C^{2,3}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}}) &= \operatorname{Hom}(V \otimes \odot^2 S, V) \oplus \operatorname{Hom}(\odot^3 S, S) \end{aligned} \tag{34}$$

and the differentials are such that if $\phi = \lambda + \mu \in C^{2,1}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}})$, with $\lambda : V \to \mathfrak{so}(V)$ and $\mu : V \to \mathfrak{r}$, then

$$\partial\phi(v, w) = \lambda_v w - \lambda_w v, \qquad \partial\phi(v, s) = \tfrac{1}{2}\omega_{\lambda_v} \cdot s + \mu_v(s) \qquad \text{and} \qquad \partial\phi(s, s) = -\lambda_{[s,s]} - \mu_{[s,s]}, \tag{35}$$

and if $\psi = \alpha + \beta + \gamma + \rho \in C^{2,2}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}})$ with

$$\alpha : \Lambda^2 V \to V, \qquad \beta : V \otimes S \to S, \qquad \gamma : \odot^2 S \to \mathfrak{so}(V) \qquad \text{and} \qquad \rho : \odot^2 S \to \mathfrak{r}, \tag{36}$$

then

$$\partial\psi(v, s, s) = [v, \gamma(s, s)] - 2[s, \beta_v s] - \alpha([s, s], v) \quad \text{and} \quad \tfrac{1}{3}\partial\psi(s, s, s) = [s, \gamma(s, s)] + [s, \rho(s, s)] - \beta_{[s,s]} s. \tag{37}$$

As before, we see that $\partial : C^{2,1}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}}) \to C^{2,2}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}})$ is injective and hence we calculate $H^{2,2}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}})$ by calculating the kernel of the Spencer differential on the space of normalised cochains in $C^{2,2}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}})$: those which have $\alpha = 0$ and for which

$$\rho(s, s)^A{}_B = \tfrac{1}{6}\omega_{\mu\nu\rho}{}^{CD}\rho^{\mu\nu\rho}{}_{CD}{}^A{}_B, \tag{38}$$

where $\rho \in \Lambda^3_- V \otimes \odot^2 \Delta \otimes \mathfrak{r}$.
In Section 5 we calculate this cohomology.

4. Calculation of $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$

We now compute the Spencer cohomology group $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ corresponding to the unextended $(1,0)$ Poincaré superalgebra.
By Proposition 8, the Spencer cohomology $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ is isomorphic to the solutions $\beta : V \otimes S \to S$ and $\gamma : \odot^2 S \to \mathfrak{so}(V)$ of the following two cocycle conditions:

$$\gamma(s, s)v + 2[s, \beta_v s] = 0 \tag{39}$$

$$\tfrac{1}{2}\omega_{\gamma(s,s)} s + \beta_{[s,s]} s = 0. \tag{40}$$

The first equation will determine γ in terms of β, so that the actual variables are the coefficients of β. It pays to understand this space and to label its components. First of all, we may view β as a map $V \to \operatorname{End}(S)$, sending $v \in V$ to $\beta_v \in \operatorname{End}(S)$, where (for $v = \boldsymbol{e}_\mu$)

$$(\beta_\mu s)^B = \beta^{(0)}_\mu{}^B{}_C s^C + \tfrac{1}{2}\beta^{(2)}_{\mu\rho\sigma}{}^B{}_C \Gamma^{\rho\sigma} s^C. \tag{41}$$

We find it convenient to rename the different components of β:

$$\begin{aligned} A_\mu \delta^A{}_B &: V \to \Lambda^0 V \otimes \Lambda^2 \Delta \\ C_\mu{}^A{}_B &: V \to \Lambda^0 V \otimes \odot^2 \Delta \\ H_{\mu\rho\sigma} \delta^A{}_B &: V \to \Lambda^2 V \otimes \Lambda^2 \Delta \\ G_{\mu\rho\sigma}{}^A{}_B &: V \to \Lambda^2 V \otimes \odot^2 \Delta, \end{aligned} \tag{42}$$

so that

$$(\beta_\mu s)^B = A_\mu s^B + C_\mu{}^B{}_C s^C + \tfrac{1}{2} H_{\mu\rho\sigma} \Gamma^{\rho\sigma} s^B + \tfrac{1}{2} G_{\mu\rho\sigma}{}^B{}_C \Gamma^{\rho\sigma} s^C. \tag{43}$$

4.1. **Solving the first cocycle condition.** We take the inner product of $\boldsymbol{e}_\mu$ with the first cocycle equation (39) applied to $v = \boldsymbol{e}_\nu$ and obtain

$$0 = \gamma(s,s)_{\mu\nu} + 2\epsilon_{AB} \bar{s}^A \Gamma_\mu (\beta_\nu s)^B = \gamma(s,s)_{\mu\nu} + 2\kappa_\mu A_\nu + 2\kappa^\rho H_{\nu\mu\rho} + G_{\nu\rho\sigma AB} \omega_\mu{}^{\rho\sigma AB}. \tag{44}$$

The skew-symmetric part gives $\gamma(s,s)_{\mu\nu}$, whereas the symmetric part gives an equation for β:

$$\kappa_\mu A_\nu + \kappa_\nu A_\mu + \kappa^\rho H_{\nu\mu\rho} + \kappa^\rho H_{\mu\nu\rho} + \tfrac{1}{2} G_{\nu\rho\sigma} \omega_\mu{}^{\rho\sigma} + \tfrac{1}{2} G_{\mu\rho\sigma} \omega_\nu{}^{\rho\sigma} = 0, \tag{45}$$

where we have suppressed the $\odot^2\Delta$ indices in G and ω. This equation is true for all s and hence the terms in the two independent bilinears (the Dirac current κ and the family ω of self-dual 3-forms) are separately zero, giving two equations:

$$\kappa^\rho (\eta_{\rho\mu} A_\nu + \eta_{\rho\nu} A_\mu + H_{\mu\nu\rho} + H_{\nu\mu\rho}) = 0 \tag{46}$$

$$\omega^{\rho\sigma\tau} (\eta_{\tau\mu} G_{\nu\rho\sigma} + \eta_{\tau\nu} G_{\mu\rho\sigma}) = 0. \tag{47}$$

Abstracting κ from the first equation and contracting first with $\eta^{\mu\nu}$ and then with $\eta^{\nu\rho}$ one finds that $A = 0$ and plugging that back into the equation one finds that $H \in \Lambda^3 V$. Since ω is self-dual, it is only the antiself-dual projection of $\eta_{\tau\mu} G_{\nu\rho\sigma} + \eta_{\tau\nu} G_{\mu\rho\sigma}$ which must vanish, yielding the equation

$$\eta_{\tau\nu} G_{\mu\rho\sigma} + \eta_{\tau\mu} G_{\nu\rho\sigma} + \eta_{\rho\nu} G_{\mu\sigma\tau} + \eta_{\rho\mu} G_{\nu\sigma\tau} + \eta_{\sigma\nu} G_{\mu\tau\rho} + \eta_{\sigma\mu} G_{\nu\tau\rho} = \tfrac{1}{2} \varepsilon_{\tau\rho\sigma\nu}{}^{\phi\psi} G_{\mu\phi\psi} + \tfrac{1}{2} \varepsilon_{\tau\rho\sigma\mu}{}^{\phi\psi} G_{\nu\phi\psi}. \tag{48}$$

Contracting with $\eta^{\mu\nu}$, we find that

$$G_{[\rho\sigma\tau]} = \tfrac{1}{6} \varepsilon_{\rho\sigma\tau}{}^{\lambda\mu\nu} G_{\lambda\mu\nu} \implies G_{[\rho\sigma\tau]} \in \Lambda^3_+ V, \tag{49}$$

whereas contracting with $\eta^{\nu\sigma}$ one finds

$$-5 G_{\mu\rho\tau} + \eta_{\rho\mu} G_\sigma{}^\sigma{}_\tau - \eta_{\tau\mu} G_\sigma{}^\sigma{}_\rho = 3 G_{[\mu\rho\tau]}. \tag{50}$$

Skew-symmetrising, one finds that $G_{[\mu\rho\tau]} = 0$ and hence

$$G_{\mu\rho\tau} = \eta_{\rho\mu} \varphi_\tau - \eta_{\tau\mu} \varphi_\rho, \tag{51}$$

where $\varphi_\rho = \frac{1}{5} G_\sigma{}^\sigma{}_\rho$. Plugging this back into the equation (47) for the family of self-dual 3-forms, we see that it is identically satisfied. We arrived at the following

Proposition 9. *The solution of the first cocycle equation is*

$$\begin{aligned} (\beta_\mu s)^A &= C_\mu{}^A{}_B s^B + \tfrac{1}{2} H_{\mu\rho\sigma} \Gamma^{\rho\sigma} s^A + \varphi^{\rho A}{}_B \Gamma_{\mu\rho} s^B \\ \gamma(s,s)_{\mu\nu} &= 2\kappa^\rho H_{\rho\mu\nu} - 2\varphi^\rho{}_{AB} \omega_{\mu\nu\rho}{}^{AB}, \end{aligned} \tag{52}$$

for some $H \in \Lambda^3 V$ and $\varphi \in V \otimes \odot^2 \Delta$.

4.2. **Solving the second cocycle condition.** We now consider the second cocycle condition (40). Using that the Dirac current κ Clifford annihilates s (Lemma 2), we may rewrite this condition as follows:

$$\kappa^\rho E_\rho{}^C{}_D s^D + \tfrac{1}{6}\omega_{\mu\nu\rho}{}^{AB}\Omega^{\mu\nu\rho}{}_{AB}s^C = 0, \tag{53}$$

where

$$\begin{aligned} E_\rho{}^C{}_D &:= H_{\rho\mu\nu}\Gamma^{\mu\nu}\delta^C{}_D + C_\rho{}^C{}_D + \varphi_\rho{}^C{}_D \\ \Omega^{\mu\nu\rho}{}_{AB} &:= -3\varphi^{[\rho}{}_{AB}\Gamma^{\mu\nu]^-}. \end{aligned} \tag{54}$$

Note that we are taking the antiself-dual projection in the RHS of the last equation, that is, we have just introduced a family Ω of antiself-dual 3-forms.

We now polarise equation (53)

$$\epsilon_{AB}\bar{s}_1^A\Gamma^\rho s_2^B E_\rho{}^C{}_D s_3^D + \tfrac{1}{6}\bar{s}_1^A\Gamma_{\mu\nu\rho}s_2^B\Omega^{\mu\nu\rho}{}_{AB}s_3^C + \text{cyclic} = 0, \tag{55}$$

set $s_1 = s_2 = s$ and rearrange to arrive at

$$\left(\kappa^\rho E_\rho{}^C{}_D + \tfrac{1}{6}\omega_{\mu\nu\rho}{}^{AB}\Omega^{\mu\nu\rho}{}_{AB}\delta^C{}_D + 2\epsilon_{AD}E_\rho{}^C{}_B s^B\bar{s}^A\Gamma^\rho + \tfrac{1}{3}\Omega^{\mu\nu\rho}{}_{AD}s^C\bar{s}^A\Gamma_{\mu\nu\rho}\right)s_3^D = 0. \tag{56}$$

We may abstract s_3, keeping in mind that $\Gamma_7 s_3 = s_3$, and use the Fierz identity (13) to arrive at

$$\begin{aligned}\Big(\kappa^\rho E_\rho{}^C{}_D + \tfrac{1}{6}\omega_{\mu\nu\rho}{}^{AB}\Omega^{\mu\nu\rho}{}_{AB}\delta^C{}_D + \tfrac{1}{4}E_\rho{}^C{}_D\kappa\Gamma^\rho + \tfrac{1}{4}E_\rho{}^C{}_B\omega^B{}_D\Gamma^\rho \\ + \tfrac{1}{24}\Omega^{\mu\nu\rho\,C}{}_D\kappa\Gamma_{\mu\nu\rho} - \tfrac{1}{24}\Omega^{\mu\nu\rho}{}_{AD}\omega^{CA}\Gamma_{\mu\nu\rho}\Big)P_+ = 0.\end{aligned} \tag{57}$$

The terms in κ and ω must vanish separately, since this expression is true for all $s \in S$. The κ terms give

$$\kappa^\sigma\left(E_\sigma{}^C{}_D + \tfrac{1}{4}E_\rho{}^C{}_D\Gamma_\sigma\Gamma^\rho + \tfrac{1}{24}\Omega^{\mu\nu\rho\,C}{}_D\Gamma_\sigma\Gamma_{\mu\nu\rho}\right)P_+ = 0, \tag{58}$$

which, abstracting κ, substituting for E and Ω and simplifying, reduces to

$$\left(H_{\sigma\mu\nu}\Gamma^{\mu\nu}\delta^C{}_D + (C+3\varphi)_\sigma{}^C{}_D + \tfrac{1}{4}H^{\mu\nu\rho}\Gamma_{\mu\nu}\Gamma_\sigma\Gamma_\rho\delta^C{}_D + \tfrac{1}{4}(C+3\varphi)^{\rho\,C}{}_D\Gamma_\sigma\Gamma_\rho\right)P_+ = 0. \tag{59}$$

The terms in $\Lambda^2\Delta$ and in $\odot^2\Delta$ vanish separately, yielding the following two equations:

$$\left(H_{\sigma\mu\nu}\Gamma^{\mu\nu} + \tfrac{1}{4}H^{\mu\nu\rho}\Gamma_{\mu\nu}\Gamma_\sigma\Gamma_\rho\right)P_+ = 0 \tag{60}$$

$$(C+3\varphi)^{\sigma\,C}{}_D(\eta_{\rho\sigma} + \tfrac{1}{4}\Gamma_\rho\Gamma_\sigma)P_+ = 0. \tag{61}$$

Simplifying the first equation we arrive at

$$\tfrac{3}{4}\left(H_{\mu\nu\rho} - (\star H)_{\mu\nu\rho}\right)\Gamma^{\mu\nu}P_+ = 0 \implies H \in \Lambda^3_+V, \tag{62}$$

whereas simplifying the second equation (and omitting the $\odot^2\Delta$ indices) we arrive at

$$\tfrac{1}{4}\left(C^\sigma + 3\varphi^\sigma\right)\left(5\eta_{\rho\sigma} + \Gamma_{\rho\sigma}\right)P_+ = 0 \implies C = -3\varphi. \tag{63}$$

It remains to consider the ω terms in equation (57), but before doing so we make the following observation. Since H is self-dual, Lemma 3 says that $H\cdot s = 0$. Similarly, Lemma 2 says that $\kappa\cdot s = 0$, hence

$$\kappa^\rho H_{\rho\mu\nu}\Gamma^{\mu\nu}s = (H\cdot\kappa + \kappa\cdot H)\cdot s = 0. \tag{64}$$

Comparing with the second cocycle condition (53), we notice that H drops out of (53) and we may conclude that the H-dependent terms in the ω-dependent terms in equation (57) are identically zero.[1]

The remaining ω-dependent terms in equation (57) are given by

$$\left(\tfrac{1}{6}\omega_{\mu\nu\rho}{}^{AB}\Omega^{\mu\nu\rho}{}_{AB}\delta^C{}_D + \tfrac{1}{4}E_\rho{}^C{}_B\omega^B{}_D\Gamma^\rho - \tfrac{1}{24}\Omega^{\mu\nu\rho}{}_{AD}\omega^{CA}\Gamma_{\mu\nu\rho}\right)P_+ = 0, \tag{65}$$

where now

$$E_\rho{}^C{}_D = -2\varphi_\rho{}^C{}_D \qquad\text{and}\qquad \Omega^{\mu\nu\rho}{}_{AB} = -3\varphi^{[\rho}{}_{AB}\Gamma^{\mu\nu]^-}. \tag{66}$$

Making use of the Clifford identity

$$\Gamma^{\mu\nu}\omega^{CA}\Gamma_{\mu\nu\rho} = 4\omega^{CA}\Gamma_\rho, \tag{67}$$

[1] This also follows combinatorially by employing identities which follow from Lemma 4.

we simplify equation (65):

$$-\tfrac{1}{2}\left(\omega_{\mu\nu\rho}{}^{AB}\varphi^{\rho}{}_{AB}\Gamma^{\mu\nu}\delta^{C}{}_{D}+\varphi^{\rho\,C}{}_{B}\omega^{B}{}_{D}\Gamma_{\rho}-\varphi^{\rho}{}_{AD}\omega^{CA}\Gamma_{\rho}\right)P_{+}=0. \tag{68}$$

Since $\omega P_{+}=0$ (Lemma 3), we may replace $\omega\Gamma_{\rho}$ by the anticommutator

$$\omega\Gamma_{\rho}+\Gamma_{\rho}\omega=\omega_{\rho\mu\nu}\Gamma^{\mu\nu}, \tag{69}$$

resulting in the equation

$$-\tfrac{1}{2}\left(\omega_{\mu\nu\rho}{}^{AB}\varphi^{\rho}{}_{AB}\delta^{C}{}_{D}+\varphi^{\rho\,C}{}_{B}\omega_{\rho\mu\nu}{}^{B}{}_{D}-\varphi^{\rho}{}_{AD}\omega_{\rho\mu\nu}{}^{CA}\right)\Gamma^{\mu\nu}P_{+}=0. \tag{70}$$

The representation of $\mathfrak{so}(V)$ on Σ_{+} is faithful, so we may drop the $\Gamma^{\mu\nu}P_{+}$ and taking out some common factors, we arrive at

$$\omega_{\mu\nu\rho}{}^{AB}\varphi^{\rho}{}_{EF}\left(\delta^{E}{}_{A}\delta^{F}{}_{B}\delta^{C}{}_{D}+\delta^{F}{}_{B}\epsilon^{EC}\epsilon_{DA}-\delta^{E}{}_{A}\delta^{F}{}_{D}\delta^{C}{}_{B}\right)=0, \tag{71}$$

which can be seen to be identically zero using the identity

$$\epsilon^{EC}\epsilon_{DA}=\delta^{E}{}_{D}\delta^{C}{}_{A}-\delta^{E}{}_{A}\delta^{C}{}_{D}. \tag{72}$$

In summary, we have proved the following

Theorem 10. *There is an isomorphism of representations of* $\mathfrak{so}(V)\oplus\mathfrak{sp}(1)$

$$H^{2,2}(\mathfrak{p}_{-},\mathfrak{p})\cong\left(\Lambda^{3}_{+}V\otimes\Lambda^{2}\Delta\right)\oplus\left(V\otimes\odot^{2}\Delta\right)\ . \tag{73}$$

The cohomology class corresponding to elements $H\in\Lambda^{3}_{+}V$ *and* $\varphi\in V\otimes\odot^{2}\Delta$ *is represented by the cocycle* $\beta+\gamma\in C^{2,2}(\mathfrak{p}_{-},\mathfrak{p})$*, where*

$$\begin{aligned}(\beta_{\rho}s)^{A}&=\tfrac{1}{2}H_{\rho\mu\nu}\Gamma^{\mu\nu}s^{A}-3\varphi_{\rho}{}^{A}{}_{B}s^{B}+\varphi^{\sigma\,A}{}_{B}\Gamma_{\rho\sigma}s^{B},\\ \gamma(s,s)_{\mu\nu}&=2\kappa^{\rho}H_{\rho\mu\nu}-2\varphi^{\rho}{}_{AB}\omega_{\rho\mu\nu}{}^{AB}.\end{aligned} \tag{74}$$

5. Calculation of $H^{2,2}(\hat{\mathfrak{p}}_{-},\hat{\mathfrak{p}})$

We now compute the Spencer cohomology group $H^{2,2}(\hat{\mathfrak{p}}_{-},\hat{\mathfrak{p}})$ for the $(1,0)$ Poincaré superalgebra extended by the R-symmetry. The first cocycle condition does not change by the introduction of the R-symmetry and hence we can re-use the results of the previous calculation (see Proposition 9) and go directly to solving the second cocycle condition:

$$\tfrac{1}{4}\gamma(s,s)_{\mu\nu}\Gamma^{\mu\nu}s^{A}+\tfrac{1}{6}\omega_{\mu\nu\rho}{}^{CD}\rho^{\mu\nu\rho}{}_{CD}{}^{A}{}_{B}s^{B}+\kappa^{\mu}(\beta_{\mu}s)^{A}=0. \tag{75}$$

We again write it as

$$\kappa^{\rho}E_{\rho}{}^{A}{}_{B}s^{B}+\tfrac{1}{6}\omega_{\mu\nu\rho}{}^{CD}\Omega^{\mu\nu\rho}{}_{CD}{}^{A}{}_{B}s^{B}=0, \tag{76}$$

where

$$\begin{aligned}E_{\rho}{}^{A}{}_{B}&:=H_{\rho\mu\nu}\Gamma^{\mu\nu}\delta^{A}{}_{B}+C_{\rho}{}^{A}{}_{B}+\varphi^{\sigma\,A}{}_{B}\Gamma_{\rho\sigma}\\ \Omega^{\mu\nu\rho}{}_{CD}{}^{A}{}_{B}&:=\rho^{\mu\nu\rho}{}_{CD}{}^{A}{}_{B}-3\varphi^{[\rho}{}_{CD}\Gamma^{\mu\nu]^{-}}\delta^{A}{}_{B}.\end{aligned} \tag{77}$$

Here $\rho\in\Lambda^{3}_{-}V\otimes\odot^{2}\Delta\otimes\mathfrak{r}$ and in the last term of the RHS of the last equation, we are taking the antiself-dual projection.

Following the same procedure as in Section 4.2, we polarise and arrive at two equations for endomorphisms: one for the κ-dependent terms and one for the ω-dependent terms:

$$\left(\kappa^{\rho}E_{\rho}{}^{A}{}_{B}+\tfrac{1}{4}E_{\rho}{}^{A}{}_{B}\kappa\Gamma^{\rho}-\tfrac{1}{24}\Omega^{\mu\nu\rho}{}_{CB}{}^{AC}\kappa\Gamma_{\mu\nu\rho}\right)P_{+}=0 \tag{78}$$

$$\left(\tfrac{1}{6}\omega_{\mu\nu\rho}{}^{CD}\Omega^{\mu\nu\rho}{}_{CD}{}^{A}{}_{B}+\tfrac{1}{4}E_{\rho}{}^{A}{}_{D}\omega^{D}{}_{B}\Gamma^{\rho}-\tfrac{1}{24}\Omega^{\mu\nu\rho}{}_{CB}{}^{A}{}_{D}\omega^{DC}\Gamma_{\mu\nu\rho}\right)P_{+}=0. \tag{79}$$

Abstracting κ from the first equation we arrive at

$$\left(5E_{\rho}{}^{A}{}_{B}+E^{\sigma\,A}{}_{B}\Gamma_{\rho\sigma}-\tfrac{1}{6}\Omega^{\mu\nu\sigma}{}_{CB}{}^{AC}\Gamma_{\rho}\Gamma_{\mu\nu\sigma}\right)P_{+}=0. \tag{80}$$

Substituting for E and Ω and simplifying we end up with

$$\left(3\left(H_{\rho\mu\nu}-\widetilde{H}_{\rho\mu\nu}\right)\Gamma^{\mu\nu}\delta^A{}_B-\rho_{\rho\mu\nu\,CB}{}^{AC}\Gamma^{\mu\nu}+5(C+3\varphi)_\rho{}^A{}_B+(C+3\varphi)^{\sigma A}{}_B\Gamma_{\rho\sigma}\right)P_+=0, \tag{81}$$

where we used the notation $\widetilde{H}=\star H$. It is convenient to decompose ρ into its irreducible components relative to the R-symmetry. Lowering indices, we take $\rho\in\Lambda^3_-V\otimes\odot^2\Delta\otimes\odot^2\Delta$. Omitting the V indices but not the Δ indices, we have

$$\rho_{AB\,CD}=\xi_{ABCD}+(\zeta_{AC}\epsilon_{BD}+\zeta_{BC}\epsilon_{AD}+\zeta_{AD}\epsilon_{BC}+\zeta_{BD}\epsilon_{AC})+\theta(\epsilon_{AC}\epsilon_{BD}+\epsilon_{BC}\epsilon_{AD}), \tag{82}$$

where $\xi\in\Lambda^3_-V\otimes\odot^4\Delta$, $\zeta\in\Lambda^3_-V\otimes\odot^2\Delta$ and $\theta\in\Lambda^3_-V$. It follows that

$$\rho_{\rho\mu\nu\,CB}{}^{AC}\Gamma^{\mu\nu}=-4\zeta_{\rho\mu\nu}{}^A{}_B\Gamma^{\mu\nu}+3\theta_{\rho\mu\nu}\delta^A{}_B\Gamma^{\mu\nu}. \tag{83}$$

Plugging this into equation (81), and separating the equation into terms of different types under the R-symmetry and $\mathfrak{so}(V)$, we arrive at

$$C=-3\varphi,\qquad \zeta=0\qquad\text{and}\qquad H-\widetilde{H}=\theta. \tag{84}$$

It is convenient to decompose $H=H^++H^-$ into self-dual and antiself-dual parts. Clearly $H^-=\frac{1}{2}\theta$. As observed in Section 4.2, the self-dual part H^+ of H does not enter the second cocycle condition, so when we solve equation (79), we may replace H by $\frac{1}{2}\theta$. The φ-dependent terms in the equation are just as in Section 4.2 and, as shown there, they cancel identically. This leaves an equation only for ρ:

$$\tfrac{1}{6}\omega_{\mu\nu\rho}{}^{CD}\left(\rho^{\mu\nu\rho}{}_{CD}{}^A{}_B+\tfrac{3}{4}\theta^{\rho\sigma\tau}\delta^A{}_D\epsilon_{BC}\Gamma_{\sigma\tau}\Gamma^{\mu\nu}-\tfrac{1}{24}\rho^{\lambda\sigma\tau}{}_{CB}{}^A{}_D\Gamma^{\mu\nu\rho}\Gamma_{\lambda\sigma\tau}\right)P_+=0. \tag{85}$$

Using the decomposition (82) of ρ and the fact that $\zeta=0$, we may rewrite this equation in terms of ξ and θ:

$$\begin{split}\tfrac{1}{6}\omega_{\mu\nu\rho}{}^{CD}\Big(\xi^{\mu\nu\rho}{}_{BCD}{}^A+2\theta^{\mu\nu\rho}\epsilon_{BC}\delta^A{}_D+\tfrac{3}{4}\theta^{\rho\sigma\tau}\epsilon_{BC}\delta^A{}_D\Gamma_{\sigma\tau}\Gamma^{\mu\nu}\\-\tfrac{1}{24}\xi^{\lambda\sigma\tau}{}_{BCD}{}^A\Gamma^{\mu\nu\rho}\Gamma_{\lambda\sigma\tau}+\tfrac{1}{24}\theta^{\lambda\sigma\tau}\epsilon_{BC}\delta^A{}_D\Gamma^{\mu\nu\rho}\Gamma_{\lambda\sigma\tau}\Big)P_+=0.\end{split}$$

Breaking up into different types under the R-symmetry, we arrive at two separate equations, one for θ and one for ξ:

$$\begin{aligned}\tfrac{1}{6}\omega_{\mu\nu\rho}{}^{CD}\left(\xi^{\mu\nu\rho}{}_{BCD}{}^A-\tfrac{1}{24}\xi_{\lambda\sigma\tau\,BCD}{}^A\Gamma^{\mu\nu\rho}\Gamma^{\lambda\sigma\tau}\right)P_+&=0,\\ \tfrac{1}{6}\omega_{\mu\nu\rho}{}^A{}_B\left(2\theta^{\mu\nu\rho}+\tfrac{3}{4}\theta^\rho{}_{\sigma\tau}\Gamma^{\sigma\tau}\Gamma^{\mu\nu}+\tfrac{1}{24}\theta_{\lambda\sigma\tau}\Gamma^{\mu\nu\rho}\Gamma^{\lambda\sigma\tau}\right)P_+&=0.\end{aligned} \tag{86}$$

Each of these equations have terms in the Λ^0V and Λ^2V components of $\mathrm{End}(\Sigma_+)$, which must vanish separately. The Λ^0V component of the first equation says that

$$\tfrac{1}{4}\omega_{\mu\nu\rho}{}^{CD}\xi^{\mu\nu\rho}_{BCD}{}^A=0\implies\xi=0, \tag{87}$$

whereas the Λ^0V component of the second equation vanishes. The Λ^2V component of the second equation is

$$-\tfrac{3}{8}\omega_{\mu\nu\rho}{}^A{}_B\theta^\rho{}_{\sigma\tau}\left(\eta^{\nu\sigma}\eta^{\mu\alpha}\eta^{\tau\beta}+\tfrac{1}{4}\epsilon^{\mu\nu\sigma\tau\alpha\beta}\right)\Gamma_{\alpha\beta}P_+=0. \tag{88}$$

Since Σ_+ is a faithful $\mathfrak{so}(V)$-representation we may write this equation as

$$\tfrac{3}{16}\omega_{\mu\nu\rho}{}^A{}_B\theta^\rho{}_{\sigma\tau}\left(\eta^{\nu\sigma}\eta^{\mu\alpha}\eta^{\tau\beta}-\eta^{\nu\sigma}\eta^{\mu\beta}\eta^{\tau\alpha}+\tfrac{1}{2}\epsilon^{\mu\nu\sigma\tau\alpha\beta}\right)=0. \tag{89}$$

We now abstract ω, remembering that this projects onto the antiself-dual component of the resulting expression and arrive at:[2]

$$\eta^{\alpha[\mu}\theta^{\nu\rho]-\beta} - \eta^{\beta[\mu}\theta^{\nu\rho]-\alpha} - \tfrac{1}{2}\theta^{[\rho}{}_{\sigma\tau}\epsilon^{\mu\nu]-\sigma\tau\alpha\beta} = 0. \tag{91}$$

Expanding this out and simplifying, we arrive at the equation

$$\theta^{[\rho}{}_{\sigma\tau}\epsilon^{\mu\nu\alpha\beta]\sigma\tau} = 0. \tag{92}$$

Taking the Hodge dual of the LHS yields

$$\theta^{\rho}{}_{\sigma\tau}\epsilon^{\mu\nu\alpha\beta\sigma\tau}\epsilon_{\rho\mu\nu\alpha\beta\pi} = 4!\left(\delta^{\tau}_{\rho}\delta^{\sigma}_{\pi} - \delta^{\sigma}_{\rho}\delta^{\tau}_{\pi}\right)\theta^{\rho}{}_{\sigma\tau} = 0\,, \tag{93}$$

which is identically satisfied. In other words, we find that θ is unconstrained.
In summary, we have proved the following extension of Theorem 10:

Theorem 11. *There is an isomorphism of representations of* $\mathfrak{so}(V) \oplus \mathfrak{sp}(1)$

$$H^{2,2}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}}) \cong \left(\Lambda^3 V \otimes \Lambda^2\Delta\right) \oplus \left(V \otimes \odot^2\Delta\right)\ . \tag{94}$$

The cohomology class corresponding to elements $H \in \Lambda^3 V$ *and* $\varphi \in V \otimes \odot^2\Delta$ *is represented by the cocycle* $\beta + \gamma + \rho \in C^{2,2}(\hat{\mathfrak{p}}_-, \hat{\mathfrak{p}})$, *where*

$$\begin{aligned}
(\beta_\rho s)^A &= \tfrac{1}{2}H_{\rho\mu\nu}\Gamma^{\mu\nu}s^A - 3\varphi_\rho{}^A{}_B s^B + \varphi^{\sigma A}{}_B\Gamma_{\rho\sigma}s^B,\\
\gamma(s,s)_{\mu\nu} &= 2\kappa^\rho H_{\rho\mu\nu} - 2\varphi^\rho{}_{AB}\omega_{\rho\mu\nu}{}^{AB},\\
\rho(s,s)^A{}_B &= \tfrac{1}{3}\omega_{\mu\nu\rho}{}^A{}_B(H^{\mu\nu\rho} - \widetilde{H}^{\mu\nu\rho}),
\end{aligned} \tag{95}$$

with $\widetilde{H}$ *the Hodge dual of* H.

Note that the components β and γ in Theorems 10 and 11 coincide. In particular, this will lead to a uniform notion of a Killing spinor (see Definition 12 in §6). We also note that Theorem 11 is of a more general scope than Theorem 10, since it includes 3-forms which are not necessarily self-dual. As we will shortly see, Theorem 10 is adequate for the construction of a Killing superalgebra on a lorentzian six-dimensional spin manifold endowed with a closed self-dual 3-form and a family $\varphi \in \Omega^1(M;\mathfrak{sp}(1))$ of coclosed 1-forms, but it is precisely the introduction of R-symmetry transformations and Theorem 11 which allow to extend this construction (at least partially) to the non self-dual case.

6. The Killing superalgebra

By analogy with the results in [19,25,26] on four- and eleven-dimensional supergravities, we may read off from Theorem 10 (or Theorem 11) the form of a Killing spinor equation which we may use to identify the six-dimensional lorentzian geometries on which to construct rigidly supersymmetric field theories. Since the introduction of the R-symmetry results in relaxing the self-duality of the 3-form H, we will work in the more general case, specialising to the self-dual case if and when necessary.
We therefore start with the following definition 12. We recall that in our conventions S is an irreducible representation of $\mathrm{Spin}(V)$ of quaternionic dimension 2.

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[2] This equation for θ defines an $\mathfrak{so}(V)$-equivariant map $\Phi : \Lambda^3_- V \to \Lambda^2 V \otimes \Lambda^3_- V$. There is a one-dimensional space of such maps, spanned by the transpose of the $\mathfrak{so}(V)$ action $\mu : \mathfrak{so}(V) \otimes \Lambda^3_+ V \to \Lambda^3_+ V$, so that $\Phi = c\mu^T$ for some $c \in \mathbb{R}$. Let $\theta \in \Lambda^3_- V$ be in the kernel of this map. Then for all $L \in \mathfrak{so}(V)$ and $\Xi \in \Lambda^3_+ V$, we have

$$0 = \langle \Phi(\theta), L \otimes \Xi\rangle = c\,\langle \theta, \mu(L\otimes\Xi)\rangle = c\,\langle\theta, L\Xi\rangle = -c\,\langle L\theta, \Xi\rangle\,, \tag{90}$$

which implies that if $c \neq 0$ then θ is $\mathfrak{so}(V)$-invariant and hence $\theta = 0$. However, we will now see that $c = 0$ and hence θ remains unconstrained.

Definition 12. *Let (M, g) be a lorentzian six-dimensional spin manifold, with associated spinor bundle $\mathbb{S} \to M$ with typical fiber S. Let $H \in \Omega^3(M)$ be a 3-form and φ a 1-form on M with values in $\mathfrak{sp}(1)$. We say that a section ε of $\mathbb{S}$ is a **Killing spinor** if it obeys*

$$\mathscr{D}_X \varepsilon := \nabla_X \varepsilon - \iota_X H \cdot \varepsilon + 3\varphi(X) \cdot \varepsilon - X \wedge \varphi \cdot \varepsilon = 0 \ , \tag{96}$$

for all $X \in \mathfrak{X}(M)$.

We write $\mathfrak{X}(M) = \Gamma(TM)$ to identify vector fields with sections of the tangent bundle on M and $\mathfrak{S}(M) = \Gamma(\mathbb{S})$ to identify spinor fields with sections of the spinor bundle $\mathbb{S} \to M$.
Note that any non-zero Killing spinor is nowhere vanishing as it is parallel with respect to a connection on the spinor bundle.
In this section we investigate under which conditions such Killing spinors generate a Lie superalgebra. We know from [30] that if $\varphi = 0$ and $dH = 0$ then this is the case. In this paper we will not assume $\varphi = 0$ and give differential constraints separately on H and φ which guarantee the existence of the Killing superalgebra.
In practice, we shall work with complexified bundles and forms in what follows, although we will not mention this explicitly. In particular we note that the (complexification of the) spinor bundle $\mathbb{S}$ has a "Grassmann-like" decomposition

$$\mathbb{S} = \$_+ \otimes \mathcal{H} \ , \tag{97}$$

where $\$_+$ is the bundle of positive-chirality spinors and $\mathcal{H} = M \times \Delta \to M$ a trivial rank-two complex vector bundle. The Levi-Civita connection ∇ is easily seen to be compatible with this decomposition, that is

$$\nabla(\varepsilon_+ \otimes \zeta) = \nabla\varepsilon_+ \otimes \zeta + \varepsilon_+ \otimes \overline{\nabla}\zeta \tag{98}$$

for all $\varepsilon_+ \in \Gamma(\$_+)$ and $\zeta \in \Gamma(\mathcal{H})$, where $\overline{\nabla}$ is a flat connection on $\mathcal{H}$. We will also work with differential forms which take values in $\mathfrak{sl}(2, \mathbb{C})$ and, whenever necessary, use the Cartan-Killing form to identify the latter with its dual.

6.1. **Preliminaries.** We collect here a series of auxiliary differential and algebraic relations, which will be needed in the proof of the main Theorems 20 and 24.
Let ε be a non-zero section of $\mathbb{S}$. It has associated the following differential forms:

- $\omega^{(1)} \in \Omega^1(M)$ given by

$$\omega^{(1)}(X_1) = (\varepsilon, X_1 \cdot \varepsilon) \ , \tag{99}$$

- a family of self-dual 3-forms $\omega^{(3)} \in \Omega^3_+(M; \mathfrak{sp}(1))$ given by

$$\omega^{(3)}_A(X_1, X_2, X_3) = (\varepsilon, (X_1 \wedge X_2 \wedge X_3) \cdot A \cdot \varepsilon) \ , \tag{100}$$

- $\omega^{(5)} \in \Omega^5(M)$ given by

$$\omega^{(5)}(X_1, \dots, X_5) = (\varepsilon, (X_1 \wedge \cdots \wedge X_5) \cdot \varepsilon) \ , \tag{101}$$

where $X_1, \dots, X_5 \in \mathfrak{X}(M)$ and $A \in \mathfrak{sp}(1)$. We note that the 1-form $\omega^{(1)}$ is the g-dual of the Dirac current $\kappa = \kappa(\varepsilon, \varepsilon) \in \mathfrak{X}(M)$ whereas $\omega^{(5)} = -\star\omega^{(1)}$. The family of self-dual 3-forms has already been introduced at a purely linear algebra level in (11). We here adorn it with a superscript, to emphasise that it is a 3-form and avoid any confusion with the other spinor bilinears.

Proposition 13. *Let* ε *be a Killing spinor on* (M, g, H, φ)*. Then*

$$\begin{aligned}
\nabla_X \omega^{(1)} &= 2\iota_X \iota_\kappa H - 2\iota_X \iota_\varphi \omega^{(3)} \ , \\
\nabla_X \omega^{(3)}_A(X_1, X_2, X_3) &= -6 \operatorname{skew} g(X, X_1) \operatorname{tr}(A\varphi(X_2))\omega^{(1)}(X_3) - 6 \operatorname{skew} \omega^{(3)}_A(\iota_X \iota_{X_1} H, X_2, X_3) \\
&\quad + 3\omega^{(3)}_{[\varphi(X),A]}(X_1, X_2, X_3) + 3 \operatorname{skew} \omega^{(3)}_{[\varphi(X_1),A]}(X_2, X_3, X) \\
&\quad - 3 \operatorname{skew} g(X, X_1)(\iota_{[\varphi,A]}\omega^{(3)})(X_2, X_3) - \omega^{(5)}(X, X_1, X_2, X_3, \operatorname{tr}(A\varphi)) \ , \\
\nabla_X \omega^{(5)} &= 2X \wedge \omega^{(1)} \wedge \star H - 2X \wedge \star(\iota_\varphi \omega^{(3)}) \ ,
\end{aligned} \tag{102}$$

for all $X, X_1, X_2, X_3 \in \mathfrak{X}(M)$ *and* $A \in \mathfrak{sp}(1)$*, where* $\operatorname{skew} = \operatorname{skew}_{X_1,X_2,X_3}$ *is skew-symmetrization on* X_1, X_2, X_3 *with weight one.*

Proof. For any Killing spinor ε and $X, Y \in \mathfrak{X}(M)$ we compute

$$\begin{aligned}
\nabla_X \omega^{(1)}(Y) &= 2\,(\varepsilon, Y \cdot \nabla_X \varepsilon) \\
&= 2\,(\varepsilon, Y \cdot \iota_X H \cdot \varepsilon) - 6\,(\varepsilon, Y \cdot \varphi(X) \cdot \varepsilon) + 2\,(\varepsilon, Y \cdot (X \wedge \varphi) \cdot \varepsilon) \\
&= 2\,(\varepsilon, \iota_Y \iota_X H \cdot \varepsilon) + 2\,(\varepsilon, Y \wedge X \wedge \varphi \cdot \varepsilon) \ ,
\end{aligned} \tag{103}$$

where last identity is a consequence of the decompositions

$$\begin{aligned}
\odot^2 S &= \Lambda^1 V \oplus (\Lambda^3_+ V \otimes \odot^2 \Delta) \ , \\
\Lambda^2 S &= (\Lambda^1 V \otimes \odot^2 \Delta) \oplus \Lambda^3_+ V \ .
\end{aligned} \tag{104}$$

This shows the first equation in (102) and applying $\star$ on both sides of it readily yields the last equation too. Similarly, for all $X, X_1, X_2, X_3 \in \mathfrak{X}(M)$ and $A \in \mathfrak{sp}(1)$, we compute

$$\begin{aligned}
\nabla_X \omega^{(3)}_A(X_1, X_2, X_3) &= 2\,(\varepsilon, (X_1 \wedge X_2 \wedge X_3) \cdot A \cdot \nabla_X \varepsilon) \\
&= 2\,(\varepsilon, (X_1 \wedge X_2 \wedge X_3) \cdot \iota_X H \cdot A \cdot \varepsilon) - 6\,(\varepsilon, (X_1 \wedge X_2 \wedge X_3) \cdot A \cdot \varphi(X) \cdot \varepsilon) \\
&\quad + 2\,(\varepsilon, (X_1 \wedge X_2 \wedge X_3) \cdot A \cdot (X \wedge \varphi) \cdot \varepsilon) \\
&= 2\mathfrak{S}\,(\varepsilon, X_1 \wedge X_2 \wedge \iota_{X_3} \iota_X H \cdot A \cdot \varepsilon) - 3\,(\varepsilon, (X_1 \wedge X_2 \wedge X_3) \cdot [A, \varphi(X)] \cdot \varepsilon) \\
&\quad + (\varepsilon, (X_1 \wedge X_2 \wedge X_3 \wedge X \wedge \operatorname{tr}(A\varphi)) \cdot \varepsilon) + \mathfrak{S}\,(\varepsilon, \iota_{X_1}(X \wedge [A, \varphi]) \wedge X_2 \wedge X_3 \cdot \varepsilon) \\
&\quad + \mathfrak{S} \iota_{X_1} \iota_{X_2}(X \wedge \operatorname{tr}(A\varphi))\,(\varepsilon, X_3 \cdot \varepsilon)
\end{aligned} \tag{105}$$

where $\mathfrak{S} = \mathfrak{S}_{X_1,X_2,X_3}$ is the cyclic sum and in the last step we repeatedly used the decomposition (104) and the identity

$$2AB = [A, B] + \operatorname{tr}(AB) \operatorname{Id} \ , \tag{106}$$

which holds for all traceless complex 2×2-matrices. The second equation in (102) is just (105) combined with definitions (99)-(101). □

Corollary 14. *Let* ε *be a Killing spinor on* (M, g, H, φ)*. Then*

$$\begin{aligned}
d\omega^{(1)} &= 4\iota_\kappa H - 4\iota_\varphi \omega^{(3)} \ , \\
d\omega^{(3)}_A(X_0, X_1, X_2, X_3) &= -24 \operatorname{Skew} \omega^{(3)}_A(\iota_{X_0} \iota_{X_1} H, X_2, X_3) - 4(\iota_{\operatorname{tr}(A\varphi)} \omega^{(5)})(X_0, X_1, X_2, X_3) \ , \\
d\omega^{(5)} &= 0 \ ,
\end{aligned} \tag{107}$$

for all $X_0, \ldots, X_3 \in \mathfrak{X}(M)$ *and* $A \in \mathfrak{sp}(1)$*, where* $\operatorname{Skew} = \operatorname{Skew}_{X_0,X_1,X_2,X_3}$ *is skew-symmetrization on* $X_0, \ldots, X_3$ *with weight one. In particular the Dirac current* κ *is a Killing vector field.*

Proof. By Proposition 13 we have that $\nabla\omega^{(1)} = \frac{1}{2}d\omega^{(1)}$. In other words $\omega^{(1)}$ is a coclosed conformal Killing 1-form, κ a Killing vector field and $d\omega^{(5)} = -d\star\omega^{(1)} = 0$. It remains to compute

$$\begin{aligned} d\omega^{(3)}_A(X_0,X_1,X_2,X_3) &= 4\,\mathrm{Skew}\,\nabla_{X_0}\omega^{(3)}_A(X_1,X_2,X_3)\\ &= -24\,\mathrm{Skew}\,\omega^{(3)}_A(\imath_{X_0}\imath_{X_1}H,X_2,X_3) + 12\,\mathrm{Skew}\,\omega^{(3)}_{[\varphi(X_0),A]}(X_1,X_2,X_3)\\ &\quad - 12\,\mathrm{Skew}\,\omega^{(3)}_{[\varphi(X_0),A]}(X_1,X_2,X_3) - 4\omega^{(5)}(X_0,X_1,X_2,X_3,\mathrm{tr}(A\varphi))\\ &= -24\,\mathrm{Skew}\,\omega^{(3)}_A(\imath_{X_0}\imath_{X_1}H,X_2,X_3) - 4(\imath_{\mathrm{tr}(A\varphi)}\omega^{(5)})(X_0,X_1,X_2,X_3)\ , \end{aligned} \tag{108}$$

completing the proof. $\square$

To proceed further, we shall need some algebraic facts on partial and full skew-symmetrisations of terms of the form $\alpha(\imath_{X_0}\imath_{X_1}\beta,X_2,X_3)$, where $\alpha,\beta\in\Omega^3(M)$. Such terms appear in (102) and (107), and they will play a crucial role towards the proof of Theorems 20 and 24.

Lemma 15. *Let $\alpha,\beta\in\Omega^3(M)$ and consider the associated 4-form $[\alpha\cdot\beta]_4\in\Omega^4(M)$ given by*

$$[\alpha\cdot\beta]_4\,(X_0,\dots,X_3) = \mathrm{Skew}\,\alpha(\imath_{X_0}\imath_{X_1}\beta,X_2,X_3)\ , \tag{109}$$

where $X_0,\dots,X_3\in\mathfrak{X}(M)$ and $\mathrm{Skew}=\mathrm{Skew}_{X_0,X_1,X_2,X_3}$ is skew-symmetrisation on $X_0,\dots,X_3$ with weight one. Then

- *(i)* $[\alpha\cdot\beta]_4 = [\beta\cdot\alpha]_4$ *for all $\alpha,\beta\in\Omega^3(M)$;*
- *(ii)* $[\alpha\cdot\beta]_4 = 0$ *if both forms are self-dual (or antiself-dual).*

Proof. The first claim follows directly from a simple computation. The second claim is also immediate, since we have the decomposition

$$\Lambda^3_\pm V\otimes\Lambda^3_\pm V = (\Lambda^3_\pm V\otimes\Lambda^3_\pm V)_0\oplus(V\otimes\Lambda^3_\pm V)_0\oplus\odot^2_0 V \tag{110}$$

into irreducible $\mathfrak{so}(V)$-modules and therefore any $\mathfrak{so}(V)$-equivariant map from $\Lambda^3_\pm V\otimes\Lambda^3_\pm V$ to $\Lambda^4 V$ is necessarily trivial. $\square$

Proposition 16. *Let ε be a Killing spinor on (M,g,H,φ). Then*

$$d\imath_\kappa H = d\imath_\varphi\omega^{(3)} \tag{111}$$

and $\mathscr{L}_\kappa\omega^{(1)} = \mathscr{L}_\kappa\omega^{(5)} = 0$. If $H\in\Omega^3(M)$ is self-dual then $\mathscr{L}_\kappa\omega^{(3)} = 0$ too.

Proof. Equation (111) follows by applying the exterior derivative to both sides of the first identity in (107). We recall that κ is a Killing vector field by Corollary 14, whence $\mathscr{L}_\kappa\omega^{(1)} = 0$ and

$$\begin{aligned} \mathscr{L}_\kappa\omega^{(5)} &= -\mathscr{L}_\kappa\star\omega^{(1)}\\ &= -\star\mathscr{L}_\kappa\omega^{(1)} = 0\ . \end{aligned} \tag{112}$$

Now $d\omega^{(3)}_A = -4\imath_{\mathrm{tr}(A\varphi)}\omega^{(5)}$ if H is self-dual, by Corollary 14 and Lemma 15. Furthermore $\imath_\kappa\omega^{(3)}_A = 0$ for all $A\in\mathfrak{sp}(1)$ by Lemma 6. It then follows

$$\begin{aligned} \mathscr{L}_\kappa\omega^{(3)}_A &= \imath_\kappa d\omega^{(3)}_A\\ &= 4\imath_{\mathrm{tr}(A\varphi)}\imath_\kappa\omega^{(5)} = 0\ , \end{aligned} \tag{113}$$

for all $A\in\mathfrak{sp}(1)$. $\square$

Lemma 17. *Let $\alpha,\beta\in\Omega^3_+(M)$ be self-dual 3-forms, with β nowhere vanishing. Let us assume that there exists a nowhere vanishing null vector field $N\in\mathfrak{X}(M)$ with the property that*

$$\imath_N\alpha = \imath_N\beta = 0 \tag{114}$$

and

$$\mathrm{skew}\,\alpha(\imath_X\imath_{X_1}\beta,X_2,X_3) = 0 \tag{115}$$

for all $X, X_1, X_2, X_3 \in \mathfrak{X}(M)$*, where* $\mathrm{skew} = \mathrm{skew}_{X_1,X_2,X_3}$ *is skew-symmetrization on* X_1, X_2, X_3 *with weight one. Then* $\alpha = f\beta$ *for some* $f \in \mathscr{C}^\infty(M)$*.*

Proof. It is enough to establish the claim pointwise. We fix $p \in M$ and a Witt basis $(\boldsymbol{e}_+, \boldsymbol{e}_-, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_4)$ of T_pM, which we use to identify T_pM with V and $N|_p$ with $\boldsymbol{e}_+$. We then write

$$V = \mathbb{R}\boldsymbol{e}_+ \oplus \mathbb{R}\boldsymbol{e}_- \oplus E\ , \tag{116}$$

where $E = \langle \boldsymbol{e}_1, \ldots, \boldsymbol{e}_4 \rangle$ is 4-dimensional euclidean. It follows from (114) that $\alpha = e_+ \wedge \widetilde{\alpha}$ and $\beta = e_+ \wedge \widetilde{\beta}$, for some antiself-dual forms $\widetilde{\alpha}, \widetilde{\beta} \in \Lambda^2_- E$. (A different choice of orientations would result in $\widetilde{\alpha}, \widetilde{\beta} \in \Lambda^2_+ E$.) Now, under the isomorphism $\Lambda^2 E \cong \mathfrak{so}(4)$, the module $\Lambda^2_- E$ gets identified with the ideal $\mathfrak{so}_-(3)$ of

$$\mathfrak{so}(4) = \mathfrak{so}_+(3) \oplus \mathfrak{so}_-(3) \tag{117}$$

and the Lie brackets on $\mathfrak{so}_-(3)$ with the skew-symmetric operation $[-,-] : \Lambda^2_- E \otimes \Lambda^2_- E \to \Lambda^2_- E$ given by

$$[\widetilde{\alpha}, \widetilde{\beta}](X_1, X_2) = \widetilde{\alpha}(\imath_{X_1}\widetilde{\beta}, X_2) - \widetilde{\alpha}(\imath_{X_2}\widetilde{\beta}, X_1)\ , \tag{118}$$

where $\widetilde{\alpha}, \widetilde{\beta} \in \Lambda^2_- E$ and $X_1, X_2 \in E$.
Equation (115) with $X = X_3 = \boldsymbol{e}_-$ leads to $[\widetilde{\alpha}, \widetilde{\beta}] = 0$ and our claim follows from the fact that the centraliser of any non-zero element in $\mathfrak{so}_-(3)$ is always 1-dimensional. □

6.2. **The Killing superalgebra. Case of self-dual** 3**-form.** Let (M, g, H, φ) be a six-dimensional lorentzian spin manifold (M, g) with spinor bundle $\mathbb{S}$ which is, in addition, endowed with a self-dual 3-form H and a 1-form φ on M with values in $\mathfrak{sp}(1)$. In this section we shall construct a Lie superalgebra $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ naturally associated with (M, g, H, φ), under appropriate conditions on H and φ.
Set

$$\begin{aligned} \mathfrak{k}_{\bar{0}} &= \{X \in \mathfrak{X}(M) \mid \mathscr{L}_X g = \mathscr{L}_X H = \mathscr{L}_X \varphi = 0\}\ , \\ \mathfrak{k}_{\bar{1}} &= \{\varepsilon \in \mathfrak{S}(M) \mid \mathscr{D}_X \varepsilon = 0 \ \text{ for all } \ X \in \mathfrak{X}(M)\}\ , \end{aligned} \tag{119}$$

where $\mathscr{D}$ is the spinor connection introduced in Definition 12. We consider the operation $[-,-] : \mathfrak{k} \otimes \mathfrak{k} \to \mathfrak{k}$ compatible with the parity of $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ and determined by the following maps:

- $[-,-] : \mathfrak{k}_{\bar{0}} \otimes \mathfrak{k}_{\bar{0}} \to \mathfrak{k}_{\bar{0}}$ is the usual commutator of vector fields,
- $[-,-] : \mathfrak{k}_{\bar{1}} \otimes \mathfrak{k}_{\bar{1}} \to \mathfrak{k}_{\bar{0}}$ is a symmetric map, with $[\varepsilon, \varepsilon] = \kappa(\varepsilon, \varepsilon)$ given by the Dirac current of $\varepsilon \in \mathfrak{k}_{\bar{1}}$, and
- $[-,-] : \mathfrak{k}_{\bar{0}} \otimes \mathfrak{k}_{\bar{1}} \to \mathfrak{k}_{\bar{1}}$ is the spinorial Lie derivative of Lichnerowicz and Kosmann (see [31]).

The fact that this operation actually takes values in $\mathfrak{k}$ is a consequence of the main Theorem 20 below, where we show that $[-,-]$ is the bracket of a Lie superalgebra structure on $\mathfrak{k}$. Assuming that result for the moment we make the following.

Definition 18. *The pair* $(\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}, [-,-])$ *is called the* Killing superalgebra *associated with* (M, g, H, φ).

Let us briefly recall the main properties of the spinorial Lie derivative, see [31] and also, e.g., [32]. The Lie derivative of a spinor field ε along a Killing vector field X is defined by $\mathscr{L}_X \varepsilon = \nabla_X \varepsilon + \sigma(A_X)\varepsilon$, where $\sigma : \mathfrak{so}(TM) \to \mathrm{End}(\mathbb{S})$ is the spin representation and $A_X = -\nabla X \in \mathfrak{so}(TM)$. It enjoys the following basic properties, for all Killing vectors X, Y, spinors ε, functions f and vector fields Z:

(i) $\mathscr{L}_X$ is a derivation:

$$\mathscr{L}_X(f\varepsilon) = X(f)\varepsilon + f\mathscr{L}_X\varepsilon\ ; \tag{120}$$

(ii) $X \mapsto \mathscr{L}_X$ is a representation of the Lie algebra of Killing vector fields:

$$\mathscr{L}_X(\mathscr{L}_Y\varepsilon) - \mathscr{L}_Y(\mathscr{L}_X\varepsilon) = \mathscr{L}_{[X,Y]}\varepsilon\ ; \tag{121}$$

(iii) $\mathscr{L}_X$ is compatible with Clifford multiplication:

$$\mathscr{L}_X(Z \cdot \varepsilon) = [X, Z] \cdot \varepsilon + Z \cdot \mathscr{L}_X\varepsilon\ ; \tag{122}$$

(iv) $\mathscr{L}_X$ is compatible with the Levi-Civita connection:

$$\mathscr{L}_X(\nabla_Z \varepsilon) = \nabla_{[X,Z]}\varepsilon + \nabla_Z(\mathscr{L}_X \varepsilon) \ . \tag{123}$$

Using property (iii), it is not difficult to see that the Dirac current is equivariant under the action of Killing vector fields, namely that

$$[X, \kappa(\varepsilon,\varepsilon)] = 2\kappa(\mathscr{L}_X \varepsilon, \varepsilon) \ , \tag{124}$$

for any Killing vector X and spinor ε. It is also clear from basic properties of Lie derivatives of vector fields that $[\mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{0}}] \subset \mathfrak{k}_{\bar{0}}$ and that for any $X \in \mathfrak{k}_{\bar{0}}$ and $Z \in \mathfrak{X}(M)$ we have

$$[\mathscr{L}_X, \mathscr{D}_Z] = \mathscr{D}_{[X,Z]}, \tag{125}$$

since $\mathscr{D}$ depends solely on the data (g, H, φ), which is preserved by $X \in \mathfrak{k}_{\bar{0}}$. This shows that $[\mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{1}}] \subset \mathfrak{k}_{\bar{1}}$ or, in other words, that the Lie algebra $\mathfrak{k}_{\bar{0}}$ acts on $\mathfrak{k}_{\bar{1}}$ via the spinorial Lie derivative. It is clear after a moment's thought that there are still conditions to be satisfied in order for $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ to be a Lie superalgebra:

(1) $\kappa(\varepsilon,\varepsilon) \in \mathfrak{k}_{\bar{0}}$, and
(2) $\mathscr{L}_{\kappa(\varepsilon,\varepsilon)}\varepsilon = 0$,

for all $\varepsilon \in \mathfrak{k}_{\bar{1}}$. The second equation is equivalent to the component of the Jacobi identity for $\mathfrak{k}$ with three odd elements. The rest of this section will be devoted to investigating (1)-(2).

We have already established in Corollary 14 that $\kappa(\varepsilon,\varepsilon)$ is a Killing vector. The following result provides a more suggestive interpretation of this fact and the Jacobi identity for three odd elements, in terms of the Spencer complex considered in §4.

Proposition 19. *The first and second cocycle conditions of the Spencer complex are equivalent to $\kappa = \kappa(\varepsilon,\varepsilon)$ being a Killing vector and $\mathscr{L}_\kappa \varepsilon = 0$, for all $\varepsilon \in \mathfrak{k}_{\bar{1}}$.*

Proof. Recall the cocycle conditions (39)-(40). For all $\varepsilon \in \mathfrak{k}_{\bar{1}}$, $Z \in \mathfrak{X}(M)$, we compute

$$\begin{aligned}\nabla_Z \kappa &= \nabla_Z \kappa(\varepsilon,\varepsilon)\\ &= 2\kappa(\nabla_Z \varepsilon, \varepsilon) = 2\kappa(\beta_Z \varepsilon, \varepsilon)\\ &= -\gamma(\varepsilon,\varepsilon)Z \ ,\end{aligned} \tag{126}$$

which says that κ is a Killing vector, since $\gamma(\varepsilon,\varepsilon)$ is a section of $\mathfrak{so}(TM)$. Similarly

$$\begin{aligned}\mathscr{L}_\kappa \varepsilon &= \nabla_\kappa \varepsilon - \sigma(\nabla\kappa)\varepsilon\\ &= \beta_\kappa \varepsilon + \sigma(\gamma(\varepsilon,\varepsilon))\varepsilon\\ &= 0 \ ,\end{aligned} \tag{127}$$

and the proposition is proved. □

We are now ready to prove the following.

Theorem 20. *Let (M, g) be a lorentzian six-dimensional spin manifold, with associated spinor bundle $\mathbb{S} \to M$ with typical fiber S. Let $H \in \Omega^3_+(M)$ be a self-dual 3-form and φ a 1-form on M with values in $\mathfrak{sp}(1)$. If*

- $dH = 0$, *and*
- $d^\star \varphi = 0$,

then there exists a natural structure of Lie superalgebra on the direct sum $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ of the spaces (119), *called the Killing superalgebra of* (M, g, H, φ).

Proof. It remains only to show that

$$\mathscr{L}_\kappa H = \mathscr{L}_\kappa \varphi = 0 \ , \tag{128}$$

where $\kappa = \kappa(\varepsilon, \varepsilon)$ is the Dirac current of a Killing spinor $\varepsilon \in \mathfrak{k}_{\bar{1}}$. We recall that $d\omega_A^{(3)} = -4\iota_{\mathrm{tr}(A\varphi)}\omega^{(5)}$ by self-duality of H, Corollary 14 and Lemma 15. It follows that

$$\begin{aligned} d\omega_A^{(3)} &= 4\iota_{\mathrm{tr}(A\varphi)} \star \omega^{(1)} \\ &= 4 \star (\omega^{(1)} \wedge \mathrm{tr}(A\varphi)) \\ &= -4\iota_\kappa(\star\,\mathrm{tr}(A\varphi)) \end{aligned} \tag{129}$$

and applying the exterior derivative to both sides yields

$$\begin{aligned} 0 &= d\iota_\kappa(\star\,\mathrm{tr}(A\varphi)) = d\iota_\kappa(\star\,\mathrm{tr}(A\varphi)) + \iota_\kappa(d(\star\,\mathrm{tr}(A\varphi))) \\ &= \mathscr{L}_\kappa \star \mathrm{tr}(A\varphi) = \star\mathscr{L}_\kappa\,\mathrm{tr}(A\varphi)\ , \end{aligned} \tag{130}$$

for all $A \in \mathfrak{sp}(1)$, or, in other words, $\mathscr{L}_\kappa\varphi = 0$. We now turn to prove $\mathscr{L}_\kappa H = 0$, which is slightly more involved.

First, we use (111) and $\iota_\kappa\omega^{(3)} = 0$ to compute

$$\begin{aligned} \iota_\kappa(\mathscr{L}_\kappa H) &= \iota_\kappa(d\iota_\kappa H) = \iota_\kappa(d\iota_\varphi\omega^{(3)}) \\ &= \mathscr{L}_\kappa\iota_\varphi\omega^{(3)} = \iota_{[\kappa,\varphi]}\omega^{(3)} + \iota_\varphi\mathscr{L}_\kappa\omega^{(3)} \\ &= 0\ , \end{aligned} \tag{131}$$

where the last identity follows from $\mathscr{L}_\kappa\varphi = \mathscr{L}_\kappa\omega^{(3)} = 0$. Hence, we have self-dual forms $\mathscr{L}_\kappa H$ and $\omega_A^{(3)}$ which vanish when evaluated on κ. Moreover, using that κ is a Killing vector, we have for all $X \in \mathfrak{X}(M)$:

$$\begin{aligned} 0 &= \nabla_X(\mathscr{L}_\kappa\omega_A^{(3)}) = \mathscr{L}_\kappa(\nabla_X\omega_A^{(3)}) - \nabla_{[\kappa,X]}\omega_A^{(3)} \\ &= -6\,\mathrm{skew}\,\omega_A^{(3)}(\iota_X\iota_{X_1}\mathscr{L}_\kappa H, X_2, X_3), \end{aligned} \tag{132}$$

where the last identity follows from a direct computation using the expression (102) of $\nabla\omega_A^{(3)}$, $\mathscr{L}_\kappa\varphi = 0$ and $\mathscr{L}_\kappa\omega^{(1)} = \mathscr{L}_\kappa\omega_A^{(3)} = \mathscr{L}_\kappa\omega^{(5)} = 0$, see Proposition 16. Here skew is, as usual, skew-symmetrisation on X_1, X_2, X_3 with weight one.

Now, let us assume for a contradiction that $\mathscr{L}_\kappa H$ is (locally) non-zero. Then Lemma 17 applies with

$$\alpha = \omega_A^{(3)}\ , \qquad \beta = \mathscr{L}_\kappa H\ , \qquad N = \kappa\ , \tag{133}$$

and $\omega_A^{(3)} = f_A\mathscr{L}_\kappa H$ for some (locally defined) function f_A, for all $A \in \mathfrak{sp}(1)$. This implies that the 3-forms $\omega_A^{(3)}$, $A \in \mathfrak{sp}(1)$, are pairwise linearly dependent at all points $p \in M$, which is absurd by Lemma 5.

The theorem is proved. □

6.3. **The Killing superalgebra. Case of H not necessarily self-dual.** This section will be devoted to constructing the Killing superalgebra when the 3-form is not necessarily self-dual. As we have seen in Theorem 11, it is precisely the introduction of the R-symmetry which allowed us to relax the self-duality assumption. We will therefore consider a six-dimensional (connected) lorentzian spin manifold (M, g) with spinor bundle $\mathbb{S} = \$_+ \otimes \mathcal{H}$, endowed with a 3-form H and a 1-form φ with values in $\mathfrak{sp}(1)$.

It will turn out that the existence of a Killing superalgebra $\hat{\mathfrak{k}} = \hat{\mathfrak{k}}_{\bar{0}} \oplus \hat{\mathfrak{k}}_{\bar{1}}$ extended by R-symmetries depends not just on some constraints on H, φ but also on an algebraic identity relating φ with the R-symmetries. Due to this, we will ultimately restrict our analysis to the case where $\varphi = 0$ (see Theorem 24).

To make contact with the notation of Theorem 11, it is convenient to introduce the bundle morphism

$$\begin{aligned} &\rho : \Lambda^3_+ T^*M \otimes \mathfrak{sp}(\mathcal{H}) \to \mathfrak{sp}(\mathcal{H})\ , \\ &\rho(\omega) = 4g(\omega, H^-)\ , \end{aligned} \tag{134}$$

where $\mathfrak{sp}(\mathcal{H}) = M \times \mathfrak{sp}(\Delta) \to M$ is the trivial rank-three subbundle with fiber $\mathfrak{sp}(\Delta)$ of the bundle of endomorphisms of $\mathbb{S}$ and $H = H^+ + H^-$ the decomposition of H into self-dual and antiself-dual components.

We set

$$\begin{aligned}
&\hat{\mathfrak{k}}_{\bar{0}} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{R}\,, \quad \text{where}\\
&\mathfrak{k}_{\bar{0}} = \{X \in \mathfrak{X}(M) \mid \mathscr{L}_X g = \mathscr{L}_X H = \mathscr{L}_X \varphi = 0\}\,,\\
&\mathfrak{R} = \{R \in \Gamma(\mathfrak{sp}(\mathcal{H})) \mid \mathscr{D}_X R = 0 \ \text{ for all } \ X \in \mathfrak{X}(M)\}\,,
\end{aligned} \tag{135}$$

and

$$\hat{\mathfrak{k}}_{\bar{1}} = \mathfrak{k}_{\bar{1}} = \{\varepsilon \in \mathfrak{S}(M) \mid \mathscr{D}_X \varepsilon = 0 \ \text{ for all } \ X \in \mathfrak{X}(M)\}\,. \tag{136}$$

Here $\mathscr{D}$ is, as usual, the spinor connection introduced in Definition 12. We note that $\mathfrak{R}$ consists of all the $\mathscr{D}$-parallel R-symmetry transformations – we will expand on this condition later on in Proposition 22. We consider the operation $[-,-]$ on $\hat{\mathfrak{k}} = \hat{\mathfrak{k}}_{\bar{0}} \oplus \hat{\mathfrak{k}}_{\bar{1}}$ determined by the usual commutator of vector fields, the Lichnerowicz-Kosmann spinorial Lie derivative and the following maps:

- $[-,-] : \mathfrak{R} \otimes \hat{\mathfrak{k}}_{\bar{1}} \to \hat{\mathfrak{k}}_{\bar{1}}$ is the natural action of a R-symmetry transformation on spinor fields;
- $[-,-] : \mathfrak{k}_{\bar{0}} \otimes \mathfrak{R} \to \mathfrak{R}$ is given by $[X, R] = \mathscr{L}_X R$, for all $X \in \mathfrak{k}_{\bar{0}}$, $R \in \mathfrak{R}$;
- $[-,-] : \mathfrak{R} \otimes \mathfrak{R} \to \mathfrak{R}$ is the commutator of two endomorphisms of the spinor bundle;
- $[-,-] : \hat{\mathfrak{k}}_{\bar{1}} \otimes \hat{\mathfrak{k}}_{\bar{1}} \to \hat{\mathfrak{k}}_{\bar{0}}$ is the symmetric map given by

$$[\varepsilon, \varepsilon] = (\kappa(\varepsilon, \varepsilon), \rho(\omega^{(3)}(\varepsilon, \varepsilon)))\,, \tag{137}$$

  where $\varepsilon \in \hat{\mathfrak{k}}_{\bar{1}}$, with associated Dirac current $\kappa(\varepsilon, \varepsilon)$ and family of self-dual 3-forms (100).

The fact that these maps actually takes values in $\hat{\mathfrak{k}}$ and define the structure of a Lie superalgebra on it depends on appropriate conditions on H and φ, which we will now start to detail.

Proposition 21. *The maps just introduced define a Lie superalgebra structure on $\hat{\mathfrak{k}}$ if and only if*

$$\begin{aligned}
\mathscr{L}_{\kappa(\varepsilon,\varepsilon)} H = \mathscr{L}_{\kappa(\varepsilon,\varepsilon)} \varphi = \mathscr{L}_{\kappa(\varepsilon,\varepsilon)} R = 0\,,\\
\mathscr{D}_X(\rho(\omega^{(3)}(\varepsilon, \varepsilon))) = 0\,,
\end{aligned} \tag{138}$$

for all $\varepsilon \in \hat{\mathfrak{k}}_{\bar{1}}$, $R \in \mathfrak{R}$ and $X \in \mathfrak{X}(M)$.

Proof. We first verify $[\hat{\mathfrak{k}}, \hat{\mathfrak{k}}] \subset \hat{\mathfrak{k}}$:

- $[\mathfrak{k}_{\bar{0}}, \mathfrak{k}_{\bar{0}}] \subset \mathfrak{k}_{\bar{0}}$, $[\mathfrak{R}, \mathfrak{R}] \subset \mathfrak{R}$ and $[\mathfrak{k}_{\bar{0}}, \hat{\mathfrak{k}}_{\bar{1}}] \subset \hat{\mathfrak{k}}_{\bar{1}}$ are straightforward;
- $[\mathfrak{k}_{\bar{0}}, \mathfrak{R}] \subset \mathfrak{R}$. We first remark that the Lichnerowicz-Kosmann Lie derivative acts trivially on any constant (=$\overline{\nabla}$-parallel, cf. the beginning of §6) section of $\mathfrak{sp}(\mathcal{H})$. Hence $\mathscr{L}_X(\Gamma(\mathfrak{sp}(\mathcal{H}))) \subset \Gamma(\mathfrak{sp}(\mathcal{H}))$ for any $X \in \mathfrak{k}_{\bar{0}}$ and the desired inclusion follows from

$$\begin{aligned}
[\mathscr{L}_X R, \mathscr{D}_Y] &= [[\mathscr{L}_X, R], \mathscr{D}_Y] = [\mathscr{L}_X, [R, \mathscr{D}_Y]] + [[\mathscr{L}_X, \mathscr{D}_Y], R]\\
&= [[\mathscr{L}_X, \mathscr{D}_Y], R] = [\mathscr{D}_{[X,Y]}, R] = 0\,,
\end{aligned} \tag{139}$$

  where $R \in \mathfrak{R}$ and $Y \in \mathfrak{X}(M)$;
- $[\mathfrak{R}, \hat{\mathfrak{k}}_{\bar{1}}] \subset \hat{\mathfrak{k}}_{\bar{1}}$ follows from

$$\begin{aligned}
\mathscr{D}_Y(R(\varepsilon)) &= \mathscr{D}_Y R(\varepsilon) + R(\mathscr{D}_Y(\varepsilon))\\
&= 0\,,
\end{aligned} \tag{140}$$

  where $R \in \mathfrak{R}$, $\varepsilon \in \hat{\mathfrak{k}}_{\bar{1}}$ and $Y \in \mathfrak{X}(M)$;
- $[\hat{\mathfrak{k}}_{\bar{1}}, \hat{\mathfrak{k}}_{\bar{1}}] \subset \hat{\mathfrak{k}}_{\bar{0}}$. We already know that the Dirac current of a Killing spinor is a Killing vector field, see Corollary 14. The remaining conditions are listed in (138).

Assuming $[\hat{\mathfrak{k}}, \hat{\mathfrak{k}}] \subset \hat{\mathfrak{k}}$, we now prove that $\hat{\mathfrak{k}} = \hat{\mathfrak{k}}_{\bar{0}} \oplus \hat{\mathfrak{k}}_{\bar{1}}$ with the operation $[-,-]$ is a Lie superalgebra. It is easy to see that $\hat{\mathfrak{k}}_{\bar{0}} = \mathfrak{k}_{\bar{0}} \ltimes \mathfrak{R}$ is the Lie algebra semidirect sum of $\mathfrak{k}_{\bar{0}}$ and $\mathfrak{R}$, acting on $\hat{\mathfrak{k}}_{\bar{1}}$ via a representation of Lie algebras. It remains to show $\hat{\mathfrak{k}}_{\bar{0}}$-equivariance of (137) and the Jacobi Identity with three odd elements.

For all $X \in \mathfrak{k}_{\bar{0}}$ and $\varepsilon \in \hat{\mathfrak{k}}_{\bar{1}}$, we compare

$$\begin{aligned}[X,[\varepsilon,\varepsilon]] &= [X,\kappa(\varepsilon,\varepsilon)] + [X,\rho(\omega^{(3)}(\varepsilon,\varepsilon))] \\ &= 2\kappa(\mathscr{L}_X\varepsilon,\varepsilon) + 4\mathscr{L}_X(g(\omega^{(3)}(\varepsilon,\varepsilon),H^-)) \\ &= 2\kappa(\mathscr{L}_X\varepsilon,\varepsilon) + 4g(\mathscr{L}_X(\omega^{(3)}(\varepsilon,\varepsilon)),H^-)\end{aligned} \tag{141}$$

with

$$2[\mathscr{L}_X\varepsilon,\varepsilon] = 2k(\mathscr{L}_X\varepsilon,\varepsilon) + 8g(\omega^{(3)}(\mathscr{L}_X\varepsilon,\varepsilon),H^-)\ , \tag{142}$$

and deduce that $\mathfrak{k}_{\bar{0}}$-equivariance of (137) follows from the identity $\mathscr{L}_X(\omega^{(3)}(\varepsilon,\varepsilon)) = 2\omega^{(3)}(\mathscr{L}_X\varepsilon,\varepsilon)$. We now check this identity. For all $A \in \mathfrak{sp}(1)$, $X_1,X_2,X_3 \in \mathfrak{X}(M)$, we compute

$$\begin{aligned}\mathscr{L}_X(\omega^{(3)}_A(\varepsilon,\varepsilon))&(X_1,X_2,X_3) \\ - 2\omega^{(3)}(\mathscr{L}_X\varepsilon,\varepsilon)(X_1,X_2,X_3) &= X\,(\varepsilon,(X_1\wedge X_2\wedge X_3)\cdot A\cdot\varepsilon) - (\varepsilon,(\mathscr{L}_X X_1\wedge X_2\wedge X_3)\cdot A\cdot\varepsilon) - \cdots \\ &\quad - (\varepsilon,(X_1\wedge X_2\wedge \mathscr{L}_X X_3)\cdot A\cdot\varepsilon) - 2\,(\nabla_X\varepsilon,(X_1\wedge X_2\wedge X_3)\cdot A\cdot\varepsilon) \\ &\quad - 2\,(\sigma(A_X)\varepsilon,(X_1\wedge X_2\wedge X_3)\cdot A\cdot\varepsilon) \\ &= -2\,(\varepsilon,(X_1\wedge X_2\wedge X_3)\cdot A\cdot\sigma(A_X)\varepsilon) - (\varepsilon,(A_X(X_1)\wedge X_2\wedge X_3)\cdot A\cdot\varepsilon) \\ &\quad - \cdots - (\varepsilon,(X_1\wedge X_2\wedge A_X(X_3))\cdot A\cdot\varepsilon) \\ &= 0\end{aligned}\ ,$$

where $A_X = -\nabla X \in \mathfrak{so}(TM)$, $\sigma : \mathfrak{so}(TM) \to \mathrm{End}(\mathbb{S})$ is the spin representation and the last equation follows from standard Clifford identities and the decomposition $\Lambda^2 S = (\Lambda^1 V \otimes \odot^2\Delta) \oplus \Lambda^3_+ V$.
Now, for all $R \in \mathfrak{R}$ we consider

$$\begin{aligned}[R,[\varepsilon,\varepsilon]] - 2[[R,\varepsilon],\varepsilon] &= [R,\kappa(\varepsilon,\varepsilon)] + [R,\rho(\omega^{(3)}(\varepsilon,\varepsilon))] - 2\kappa(R(\varepsilon),\varepsilon) - 2\rho(\omega^{(3)}(R(\varepsilon),\varepsilon)) \\ &= -\mathscr{L}_{\kappa(\varepsilon,\varepsilon)}R - 2\kappa(R(\varepsilon),\varepsilon) = -\mathscr{L}_{\kappa(\varepsilon,\varepsilon)}R \\ &= 0\ ,\end{aligned} \tag{143}$$

where we used (138). This concludes the proof of $\hat{\mathfrak{k}}_{\bar{0}}$-equivariance of (137).
The Jacobi identity with three odd elements is equivalent to the second cocycle condition of the extended Spencer complex. Indeed, for all $\varepsilon \in \hat{\mathfrak{k}}_{\bar{1}}$, we have

$$\begin{aligned}[[\varepsilon,\varepsilon],\varepsilon] &= \mathscr{L}_\kappa\varepsilon + \rho(\omega)\varepsilon \\ &= \nabla_\kappa\varepsilon - \sigma(\nabla\kappa)\varepsilon + \rho(\omega)\varepsilon \\ &= \beta_\kappa\varepsilon + \sigma(\gamma(\varepsilon,\varepsilon))\varepsilon + \rho(\omega)\varepsilon \\ &= 0\ ,\end{aligned} \tag{144}$$

where $\kappa = \kappa(\varepsilon,\varepsilon)$ and $\omega = \omega^{(3)}(\varepsilon,\varepsilon)$. The proposition is proved. □

It is clear from the definition of the space $\mathfrak{R}$ and Proposition 21 that a better understanding of $\mathscr{D}$-parallel R-symmetries is required.

Proposition 22. *A R-symmetry transformation $R \in \Gamma(\mathfrak{sp}(\mathcal{H}))$ is $\mathscr{D}$-parallel if and only if*

- *it is constant, that is $\overline{\nabla}_X R = 0$ for all $X \in \mathfrak{X}(M)$, and*
- *it pointwise commutes with φ, that is*

$$[R|_p, A] = 0 \tag{145}$$

at all points $p \in M$ and for all $A \in \varphi(T_pM) \subset \mathfrak{sp}(1)$.

In particular, if $\varphi(T_pM)$ has dimension greater than or equal to 2 at some fixed point $p \in M$ then any $\mathscr{D}$-parallel R-symmetry transformation is identically zero.

Proof. We consider the decomposition of the spinor connection

$$\mathscr{D}_X \varepsilon = D_X \varepsilon + \Phi_X \varepsilon \tag{146}$$

as sum of the metric connection with skew-symmetric torsion

$$D_X Y = \nabla_X Y + 2h(X, Y), \tag{147}$$

where $g(h(X, Y), Z) = H(X, Y, Z)$, and the φ-dependent endomorphism of the spinor bundle

$$\Phi_X \varepsilon = X \cdot \varphi \cdot \varepsilon + 2\varphi \cdot X \cdot \varepsilon \ . \tag{148}$$

We note that Φ_X is a section of $\mathfrak{sp}(\mathcal{H}) \oplus (\Lambda^2 TM \otimes \mathfrak{sp}(\mathcal{H}))$.
A straightforward computation says

$$\begin{aligned} [\mathscr{D}_X, R](\varepsilon) &= [D_X, R](\varepsilon) + [\Phi_X, R](\varepsilon) \\ &= \overline{\nabla}_X R(\varepsilon) - [R, \Phi_X](\varepsilon) \ , \end{aligned} \tag{149}$$

whence the R-symmetry transformation R is $\mathscr{D}$-parallel if and only if

$$\begin{aligned} \overline{\nabla}_X R &= [R, \Phi_X] \\ &= X \cdot [R, \varphi] + 2[R, \varphi] \cdot X \\ &= 3g(X, [R, \varphi]) - X \wedge [R, \varphi] \ , \end{aligned} \tag{150}$$

for all $X \in \mathfrak{X}(M)$. Equation (150) is an identity of endomorphisms of the spinor bundle but note that the LHS is a section of $\mathfrak{sp}(\mathcal{H})$ whereas the RHS of $\mathfrak{sp}(\mathcal{H}) \oplus (\Lambda^2 TM \otimes \mathfrak{sp}(\mathcal{H}))$. Equation (150) then splits into

$$\begin{aligned} \overline{\nabla}_X R &= 3g(X, [R, \varphi]) \ , \\ X \wedge [R, \varphi] &= 0 \ , \end{aligned} \tag{151}$$

for all $X \in \mathfrak{X}(M)$, which implies $[R, \varphi] = 0$ and $\overline{\nabla}_X R = 0$. The first claim of the proposition is proved.
The last claim follows from the fact that R is constant and the centraliser of any non-zero element of $\mathfrak{sp}(1)$ is 1-dimensional. □

Corollary 23. $\mathscr{L}_X R = 0$ *for all Killing vector fields* X *and* $R \in \mathfrak{R}$.

We deduce from Propositions 21 and 22 that in general only the *decomposable* $\varphi : TM \to \mathfrak{sp}(1)$ have an associated Killing superalgebra extended by R-symmetry transformations (in the sense defined in this section) and that some additional algebraic conditions on the space of Killing spinors have to be enforced if $\varphi \neq 0$ (so that $\rho(\omega^{(3}(\varepsilon, \varepsilon))$ pointwise commutes with φ).
We will restrict to $\varphi = 0$ in what follows. A deeper understanding of the decomposable case is an interesting problem, which we leave to future work.

Theorem 24. *Let* (M, g, H) *be a lorentzian six-dimensional spin manifold endowed with a 3-form* $H \in \Omega^3(M)$ *and* $D_X Y = \nabla_X Y + 2h(X, Y)$ *the metric connection with skew-symmetric torsion defined by* $g(h(X, Y), Z) = H(X, Y, Z)$. *Let also* $H = H^+ + H^-$ *be the decomposition of* H *into self-dual and antiself-dual components. If*

- $dH = 0$ *and*
- H^- *is* D*-parallel,*

then there exists a natural structure of Lie superalgebra on the direct sum $\hat{\mathfrak{k}} = \hat{\mathfrak{k}}_{\bar{0}} \oplus \hat{\mathfrak{k}}_{\bar{1}}$ *of the spaces* (135) *and* (136). *We call it the Killing superalgebra extended by R-symmetry transformations associated to* (M, g, H).

Proof. Due to Propositions 21 and 22 and Corollary 23, it remains to show that $\mathscr{L}_{\kappa(\varepsilon,\varepsilon)} H = 0$ and that $\rho(\omega^{(3)}(\varepsilon, \varepsilon))$ is a constant section of $\mathfrak{sp}(\mathcal{H})$, for all $\varepsilon \in \hat{\mathfrak{k}}_{\bar{1}}$. We depart with

$$\begin{aligned} \mathscr{L}_{\kappa(\varepsilon,\varepsilon)} H &= d\iota_{\kappa(\varepsilon,\varepsilon)} H + \iota_{\kappa(\varepsilon,\varepsilon)} dH \\ &= d\iota_{\kappa(\varepsilon,\varepsilon)} H = 0 \ , \end{aligned} \tag{152}$$

where the last equation follows from Proposition 16 with $\varphi = 0$, and then conclude with

$$\begin{aligned}\overline{\nabla}_X(\rho(\omega^{(3)}(\varepsilon,\varepsilon))) &= 4X(g(\omega^{(3)}(\varepsilon,\varepsilon),H^-)) \\ &= 4g(D_X(\omega^{(3)}(\varepsilon,\varepsilon)),H^-) + 4g(\omega^{(3)}(\varepsilon,\varepsilon),D_X H^-) \\ &= 8g(\omega^{(3)}(D_X\varepsilon,\varepsilon),H^-) \\ &= 0\ ,\end{aligned} \tag{153}$$

which holds for all $X \in \mathfrak{X}(M)$. □

Remark. Let D (resp. D^+) be the metric connection with skew-symmetric torsion $g(h(X,Y),Z) = H(X,Y,Z)$ (resp. $g(h(X,Y),Z) = H^+(X,Y,Z)$). Then it is not difficult to see that $DH^- = D^+H^-$, so that the second condition in Theorem 24 is equivalent to H^- being D^+-parallel.

7. Killing superalgebras (alternative calculation with some indices)

Let M be a six-dimensional spin manifold equipped with a lorentzian metric g, a three-form H and an $\mathfrak{sp}(1)$-valued one-form φ. In addition, let the spinor bundle on M be equipped with a connection $\hat{\nabla}$ whose action on a positive chirality spinor field ε is defined, with respect to the basis defined in section 2, by

$$\hat{\nabla}_\mu \varepsilon^A = \nabla_\mu \varepsilon^A + C_\mu{}^A{}_B \varepsilon^B\ , \tag{154}$$

where ∇ is the Levi-Civita connection and C is a locally defined $\mathfrak{sp}(1)$-valued one-form on M. For any $\mathrm{Sp}(1)$-valued smooth function λ on M, the transformations

$$\varepsilon \mapsto \lambda\varepsilon\ , \quad C_\mu \mapsto -(\partial_\mu\lambda)\lambda^{-1} + \lambda C_\mu \lambda^{-1}\ , \tag{155}$$

imply $\hat{\nabla}_\mu\varepsilon \mapsto \lambda\hat{\nabla}_\mu\varepsilon$. Furthermore, the curvature

$$G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu + [C_\mu, C_\nu]\ , \tag{156}$$

of C has the transformation $G_{\mu\nu} \mapsto \lambda G_{\mu\nu}\lambda^{-1}$.

Now recall that $\mathfrak{S}(M)$ denotes the space of sections of the positive chirality spinor bundle on M. In terms of the data above, motivated by Theorem 11, let us call any $\varepsilon \in \mathfrak{S}(M)$ a *Killing spinor* if

$$\mathscr{D}_\mu\varepsilon^A := \hat{\nabla}_\mu\varepsilon^A - \tfrac{1}{2}H_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon^A + 3\varphi_\mu{}^A{}_B\varepsilon^B - \varphi^{\nu A}{}_B\Gamma_{\mu\nu}\varepsilon^B = 0\ . \tag{157}$$

Notice that (157) is invariant under (155) provided the background fields transform in the obvious way:

$$g_{\mu\nu} \mapsto g_{\mu\nu}\ , \quad H_{\mu\nu\rho} \mapsto H_{\mu\nu\rho}\ , \quad \varphi_\mu \mapsto \lambda\varphi_\mu\lambda^{-1}\ . \tag{158}$$

This manifest local $\mathrm{Sp}(1)$ invariance we have engineered is sometimes referred to as 'gauging the R-symmetry' in the physics literature.

Now to the construction of the Killing superalgebra. We define a Killing superalgebra $\mathfrak{k}$ to be a Lie superalgebra whose odd part $\mathfrak{k}_{\bar{1}}$ is precisely the space of Killing spinors defined by (157). The even part $\mathfrak{k}_{\bar{0}}$ must contain elements which act as endomorphisms of $\mathfrak{k}_{\bar{1}}$, so that we may assign a bracket $[\mathfrak{k}_{\bar{0}},\mathfrak{k}_{\bar{1}}] \subset \mathfrak{k}_{\bar{1}}$. There are two obvious candidates: Killing vectors (acting via the spinorial Lie derivative) and local $\mathfrak{sp}(1)$ R-symmetries. By definition, both these transformations are endomorphisms of $\mathfrak{S}(M)$. But, as we will see in a moment, to preserve $\mathfrak{k}_{\bar{1}}$ will demand some additional constraints.

Let $\mathfrak{X}(M)$ denote the space of vector fields on M and let us write the subspace of Killing vectors

$$\mathfrak{K}(M) = \{X \in \mathfrak{X}(M) \mid \mathscr{L}_X g = 0\}\ , \tag{159}$$

where, of course, $\mathscr{L}_X$ denotes the Lie derivative along X. For any $X \in \mathfrak{K}(M)$ and $\varepsilon \in \mathfrak{S}(M)$, the spinorial Lie derivative of ε along X is defined by

$$\mathscr{L}_X\varepsilon = \nabla_X\varepsilon + \tfrac{1}{4}dX^\flat\varepsilon\ . \tag{160}$$

For any $X \in \mathfrak{K}(M)$, $Y \in \mathfrak{X}(M)$ and $\Upsilon \in \Omega^\bullet(M)$, as endomorphisms of $\mathfrak{S}(M)$, we have the useful identities

$$[\mathscr{L}_X, \nabla_Y] = \nabla_{[X,Y]} , \quad [\mathscr{L}_X, \Upsilon] = \mathscr{L}_X \Upsilon , \tag{161}$$

where $[X, Y] = \mathscr{L}_X Y$ is the Lie bracket on $\mathfrak{X}(M)$.

Since $\mathscr{L}_X$ in (160) is clearly not covariant under the local $\mathrm{Sp}(1)$ transformation $\varepsilon \mapsto \lambda\varepsilon$, let us define a more appropriate gauged version:

$$\hat{\mathscr{L}}_X \varepsilon = \hat{\nabla}_X \varepsilon + \tfrac{1}{4} dX^\flat \varepsilon , \tag{162}$$

for any $X \in \mathfrak{K}(M)$ and $\varepsilon \in \mathfrak{S}(M)$, which transforms covariantly $\hat{\mathscr{L}}_X \varepsilon \mapsto \lambda \hat{\mathscr{L}}_X \varepsilon$ under (155). The associated identities in (161) become

$$[\hat{\mathscr{L}}_X, \hat{\nabla}_Y] = \hat{\nabla}_{[X,Y]} + G(X, Y) , \quad [\hat{\mathscr{L}}_X, \Upsilon] = \mathscr{L}_X \Upsilon , \tag{163}$$

for any $X \in \mathfrak{K}(M)$, $Y \in \mathfrak{X}(M)$ and $\Upsilon \in \Omega^\bullet(M)$, where G is the curvature of C from (156). Using these identities together with the definition of $\mathscr{D}$ in (157) then yields

$$[\hat{\mathscr{L}}_X, \mathscr{D}_Y]\varepsilon = \mathscr{D}_{[X,Y]}\varepsilon + G(X, Y)\varepsilon - \iota_Y(\mathscr{L}_X H)\varepsilon + 3(\hat{\mathscr{L}}_X \varphi)(Y)\varepsilon - Y \wedge (\hat{\mathscr{L}}_X \varphi)\varepsilon , \tag{164}$$

for any $X \in \mathfrak{K}(M)$, $Y \in \mathfrak{X}(M)$ and $\varepsilon \in \mathfrak{S}(M)$, where $\hat{\mathscr{L}}_X \varphi = \mathscr{L}_X \varphi + [\iota_X C, \varphi]$. Consequently, for any $X \in \mathfrak{K}(M)$ and $\varepsilon \in \mathfrak{k}_{\bar{1}}$, we see that $\hat{\mathscr{L}}_X \varepsilon \in \mathfrak{k}_{\bar{1}}$ is guaranteed provided

$$\iota_X G = 0 , \quad \mathscr{L}_X H = 0 , \quad \hat{\mathscr{L}}_X \varphi = 0 . \tag{165}$$

Henceforth we shall define

$$\mathfrak{K} = \{X \in \mathfrak{X}(M) \,|\, \mathscr{L}_X g = 0, \mathscr{L}_X H = 0, \hat{\mathscr{L}}_X \varphi = 0, \iota_X G = 0\} , \tag{166}$$

as a natural subspace of Killing vectors which preserve the background. Notice that the Bianchi identity $d^{\hat{\nabla}} G = 0$ implies that $\hat{\mathscr{L}}_X G = d^{\hat{\nabla}} \iota_X G = 0$, for any $X \in \mathfrak{K}$.

Now to the local R-symmetries. For any $\mathfrak{sp}(1)$-valued smooth function ρ on M and any $\varepsilon \in \mathfrak{S}(M)$, it is easy to verify that

$$[\mathscr{D}_\mu, \rho]\varepsilon = (\hat{\nabla}_\mu \rho + 3[\varphi_\mu, \rho] - \Gamma_{\mu\nu}[\varphi^\nu, \rho])\varepsilon . \tag{167}$$

Thus, for any $\varepsilon \in \mathfrak{k}_{\bar{1}}$, we see that $\rho\varepsilon \in \mathfrak{k}_{\bar{1}}$ is guaranteed provided

$$\hat{\nabla}_\mu \rho = 0 , \quad [\varphi_\mu, \rho] = 0 . \tag{168}$$

Henceforth we shall define

$$\mathfrak{R} = \{\rho \in C^\infty(M) \otimes \mathfrak{sp}(1) \,|\, \hat{\nabla}\rho = 0, [\varphi, \rho] = 0\} , \tag{169}$$

as a natural subspace of local R-symmetries which preserve the background.

Next, if we are to identify $\mathfrak{k}_{\bar{0}}$ with (a subspace of) $\mathfrak{K} \oplus \mathfrak{R}$, we need to define the brackets and check the Jacobi identities for $\mathfrak{k}$.

The $[\bar{0}\bar{0}\bar{0}]$ component of the Jacobi identity just says that $\mathfrak{k}_{\bar{0}}$ must be a Lie algebra.

For any $X, Y \in \mathfrak{X}(M)$, as endomorphisms of the tensor bundle on M, the commutator of Lie derivatives $\mathscr{L}_X$ and $\mathscr{L}_Y$ obeys the identity

$$[\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{L}_{[X,Y]} , \tag{170}$$

where $[X, Y] = \mathscr{L}_X Y$ is the Lie bracket on $\mathfrak{X}(M)$. Furthermore, it is easily verified that

$$[\hat{\mathscr{L}}_X, \hat{\mathscr{L}}_Y]\varphi = \hat{\mathscr{L}}_{[X,Y]}\varphi + [G(X, Y), \varphi] , \quad [\hat{\mathscr{L}}_X, \iota_Y]G = \iota_{[X,Y]}G , \tag{171}$$

for any $X, Y \in \mathfrak{X}(M)$. Thus, for any $X, Y \in \mathfrak{K}$, (170) implies $\mathscr{L}_{[X,Y]} g = 0$ and $\mathscr{L}_{[X,Y]} H = 0$, while (171) implies $\hat{\mathscr{L}}_{[X,Y]}\varphi = 0$ (using $\iota_X G = 0$) and $\iota_{[X,Y]} G = 0$ (using $\hat{\mathscr{L}}_X G = 0$ and $\iota_Y G = 0$). It follows that $\mathfrak{K}$ is indeed a Lie algebra with respect to the Lie bracket of vector fields.

For any $\rho, \rho' \in C^\infty(M) \otimes \mathfrak{sp}(1)$, as endomorphisms of $\mathfrak{S}(M)$, clearly the commutator $[\rho, \rho'] = \rho\rho' - \rho'\rho$ obeys

$$\hat{\nabla}[\rho, \rho'] = [\hat{\nabla}\rho, \rho'] + [\rho, \hat{\nabla}\rho'] , \quad [\varphi, [\rho, \rho']] = [[\varphi, \rho], \rho'] + [\rho, [\varphi, \rho']] . \tag{172}$$

Whence, any $\rho, \rho' \in \mathfrak{R}$ have $[\rho, \rho'] \in \mathfrak{R}$ and $\mathfrak{R}$ is clearly a Lie algebra with respect to the commutator of endomorphisms.

So $\mathfrak{K} \oplus \mathfrak{R}$ is certainly a Lie algebra if we define $[\mathfrak{K}, \mathfrak{R}] = 0$. Indeed, even if we had defined $[X, \rho] = \hat{\nabla}_X \rho$, for any $X \in \mathfrak{K}$ and $\rho \in \mathfrak{R}$, the condition $\hat{\nabla}\rho = 0$ in (169) would force us to take $[\mathfrak{K}, \mathfrak{R}] = 0$.

The $[\bar{0}\bar{0}\bar{1}]$ component of the Jacobi identity says that $\mathfrak{k}_{\bar{0}}$ must act on $\mathfrak{k}_{\bar{1}}$ as a $\mathfrak{k}_{\bar{0}}$-module. For any $X, Y \in \mathfrak{K}(M)$ and $\varepsilon \in \mathfrak{S}(M)$, one finds that the commutator of gauged spinorial Lie derivatives in (162) obeys the identity

$$[\hat{\mathscr{L}}_X, \hat{\mathscr{L}}_Y]\varepsilon = \hat{\mathscr{L}}_{[X,Y]}\varepsilon + G(X, Y)\varepsilon \ . \tag{173}$$

Whence, for any $X \in \mathfrak{K}$ and $\varepsilon \in \mathfrak{k}_{\bar{1}}$, the bracket

$$[X, \varepsilon] = \hat{\mathscr{L}}_X \varepsilon \ , \tag{174}$$

defines $\mathfrak{k}_{\bar{1}}$ as a $\mathfrak{K}$-module (since $\iota_X G = 0$). Moreover, for any $\rho \in \mathfrak{R}$ and $\varepsilon \in \mathfrak{k}_{\bar{1}}$, the bracket

$$[\rho, \varepsilon] = \rho\varepsilon \ , \tag{175}$$

clearly defines $\mathfrak{k}_{\bar{1}}$ as an $\mathfrak{R}$-module by restricting local $\mathfrak{sp}(1)$ endomorphisms of the spinor bundle. Finally, combining (174) and (175), we see that

$$[X, [\rho, \varepsilon]] - [\rho, [X, \varepsilon]] = [\hat{\mathscr{L}}_X, \rho]\varepsilon = (\hat{\nabla}_X \rho)\varepsilon = 0 \ , \tag{176}$$

as required, for any $X \in \mathfrak{K}$, $\rho \in \mathfrak{R}$ and $\varepsilon \in \mathfrak{k}_{\bar{1}}$ (since $\hat{\nabla}\rho = 0$).

This has established that $\mathfrak{k}_{\bar{1}}$ is indeed a representation of the Lie algebra $\mathfrak{K} \oplus \mathfrak{R}$ with respect to the action defined by (174) and (175).

In order to check the remaining Jacobi identities for $\mathfrak{k}$, we must first specify a bracket $[\mathfrak{k}_{\bar{1}}, \mathfrak{k}_{\bar{1}}] \subset \mathfrak{k}_{\bar{0}}$. Since the odd-odd bracket for $\mathfrak{k}$ is symmetric, it is sufficient to define

$$[\varepsilon, \varepsilon] = (\kappa(\varepsilon), \vartheta(\varepsilon)) \ , \tag{177}$$

for all $\varepsilon \in \mathfrak{k}_{\bar{1}}$, such that $\kappa(\varepsilon) \in \mathfrak{K}$ and $\vartheta(\varepsilon) \in \mathfrak{R}$. The bracket of two different $\varepsilon, \varepsilon' \in \mathfrak{k}_{\bar{1}}$ is then defined by polarisation:

$$[\varepsilon, \varepsilon'] = \tfrac{1}{2}([\varepsilon + \varepsilon', \varepsilon + \varepsilon'] - [\varepsilon, \varepsilon] - [\varepsilon', \varepsilon']) \ , \tag{178}$$

and it is convenient to define $\kappa(\varepsilon, \varepsilon') = \frac{1}{2}(\kappa(\varepsilon + \varepsilon') - \kappa(\varepsilon) - \kappa(\varepsilon'))$ and $\vartheta(\varepsilon, \varepsilon') = \frac{1}{2}(\vartheta(\varepsilon + \varepsilon') - \vartheta(\varepsilon) - \vartheta(\varepsilon'))$.

Guided again by Theorem 11, let us consider the following choices:

$$\kappa(\varepsilon)^\mu = \epsilon_{AB}\bar{\varepsilon}^A \Gamma^\mu \varepsilon^B \ , \quad \vartheta(\varepsilon)^{AB} = \tfrac{2}{3} H^{\mu\nu\rho}\, \bar{\varepsilon}^A \Gamma_{\mu\nu\rho} \varepsilon^B \ . \tag{179}$$

For a given $\varepsilon \in \mathfrak{k}_{\bar{1}}$, it will sometimes be convenient to drop the parenthetical ε in (179) and write $\kappa^\mu = \epsilon_{AB}\bar{\varepsilon}^A \Gamma^\mu \varepsilon^B$ and $\omega^{AB}_{\mu\nu\rho} = \bar{\varepsilon}^A \Gamma_{\mu\nu\rho} \varepsilon^B$ for the Killing spinor bilinears.

Clearly (179) defines $\kappa \in \mathfrak{X}(M)$ and $\vartheta \in C^\infty(M) \otimes \mathfrak{sp}(1)$. However, for $\kappa \in \mathfrak{K}$ and $\vartheta \in \mathfrak{R}$, we require all of the following conditions to be satisfied:

$$\mathscr{L}_\kappa g = 0 \ , \quad \mathscr{L}_\kappa H = 0 \ , \quad \hat{\mathscr{L}}_\kappa \varphi = 0 \ , \quad \iota_\kappa G = 0 \ , \quad \hat{\nabla}\vartheta = 0 \ , \quad [\varphi, \vartheta] = 0 \ . \tag{180}$$

We shall return to the important matter of checking whether these conditions are actually satisfied in a moment but first let us just assume that they are and move on to confirm the remaining Jacobi identities.

The $[\bar{0}\bar{1}\bar{1}]$ component of the Jacobi identity says that the odd-odd bracket on $\mathfrak{k}$ must define a $\mathfrak{k}_{\bar{0}}$-equivariant map $\mathfrak{k}_{\bar{1}} \otimes \mathfrak{k}_{\bar{1}} \to \mathfrak{k}_{\bar{0}}$. This means that the $\mathfrak{k}_{\bar{1}} \otimes \mathfrak{k}_{\bar{1}} \to \mathfrak{K}$ part must be $\mathfrak{K}$-equivariant and $\mathfrak{R}$-invariant (since $[\mathfrak{R}, \mathfrak{K}] = 0$) while the $\mathfrak{k}_{\bar{1}} \otimes \mathfrak{k}_{\bar{1}} \to \mathfrak{R}$ part must be $\mathfrak{R}$-equivariant and $\mathfrak{K}$-invariant (since $[\mathfrak{K}, \mathfrak{R}] = 0$).

For any $X \in \mathfrak{K}(M)$ and $\varepsilon \in \mathfrak{S}(M)$, we have the identities

$$[X, \kappa] = 2\kappa(\hat{\mathscr{L}}_X \varepsilon, \varepsilon) \ , \quad \hat{\nabla}_X \vartheta = 2\vartheta(\hat{\mathscr{L}}_X \varepsilon, \varepsilon) + \tfrac{2}{3}(\mathscr{L}_X H)_{\mu\nu\rho}\omega^{\mu\nu\rho} \ . \tag{181}$$

The first identity above guarantees that $\mathfrak{k}_{\bar{1}} \otimes \mathfrak{k}_{\bar{1}} \to \mathfrak{K}$ is $\mathfrak{K}$-equivariant. Moreover, if $X \in \mathfrak{K}$ and $\vartheta \in \mathfrak{R}$ then $\mathscr{L}_X H = 0$ and $\hat{\nabla}\vartheta = 0$, in which case the second identity in (181) says that $\mathfrak{k}_{\bar{1}} \otimes \mathfrak{k}_{\bar{1}} \to \mathfrak{R}$ is indeed $\mathfrak{K}$-invariant.

For any $\rho \in C^\infty(M) \otimes \mathfrak{sp}(1)$ and $\varepsilon \in \mathfrak{S}(M)$, we also have the identities

$$[\rho, \vartheta] = 2\vartheta(\rho\varepsilon, \varepsilon) , \quad \kappa(\rho\varepsilon, \varepsilon) = 0 . \tag{182}$$

The first identity above shows that $\mathfrak{k}_{\bar{1}} \otimes \mathfrak{k}_{\bar{1}} \to \mathfrak{R}$ is $\mathfrak{R}$-equivariant while the second identity shows that $\mathfrak{k}_{\bar{1}} \otimes \mathfrak{k}_{\bar{1}} \to \mathfrak{K}$ is $\mathfrak{R}$-invariant.
Whence, at least if $\kappa \in \mathfrak{K}$ and $\vartheta \in \mathfrak{R}$, we have shown that the $[\bar{0}\bar{1}\bar{1}]$ component of the Jacobi identity is satisfied.
The final $[\bar{1}\bar{1}\bar{1}]$ component of the Jacobi identity is equivalent (via polarisation) to the condition

$$[[\varepsilon, \varepsilon], \varepsilon] = 0 , \tag{183}$$

for all $\varepsilon \in \mathfrak{k}_{\bar{1}}$. To examine this condition more closely, it is worth noting the identity

$$\nabla_\mu \kappa_\nu = 2H_{\mu\nu\rho}\kappa^\rho - 2\varphi^\rho_{AB}\omega^{AB}_{\mu\nu\rho} , \tag{184}$$

which can be derived from the definition of κ in (179) using the Killing spinor equation (157). Using the brackets defined by (174), (175), (177) and (179), the left hand side of (183) reads

$$(\hat{\mathscr{L}}_\kappa \varepsilon + \vartheta\varepsilon)^A = \kappa^\mu \hat{\nabla}_\mu \varepsilon^A + \tfrac{1}{4}(\nabla_\mu \kappa_\nu)\Gamma^{\mu\nu}\varepsilon^A + \tfrac{1}{3}(H^{\mu\nu\rho} - \tilde{H}^{\mu\nu\rho})\omega_{\mu\nu\rho}{}^A{}_B \varepsilon^B . \tag{185}$$

We have used the identity $H^{\mu\nu\rho}\omega^{AB}_{\mu\nu\rho} = H^{\mu\nu\rho}\tilde{\omega}^{AB}_{\mu\nu\rho} = -\tilde{H}^{\mu\nu\rho}\omega^{AB}_{\mu\nu\rho}$ to identify the contribution of $\vartheta^A{}_B$ in the third term of (185) with $\rho(s,s)^A{}_B$ in Theorem 11. Moreover, notice that (184) allows us to identify the contribution of $\nabla_\mu\kappa_\nu$ in the second term of (185) with $\gamma(s,s)_{\mu\nu}$ in Theorem 11. Finally, the Killing spinor equation (157) allows us to identify the contribution of $\hat{\nabla}_\mu \varepsilon^A$ in the first term of (185) with $(\beta_\mu s)^A$ in Theorem 11. The vanishing of (185) is therefore precisely equivalent to the second cocycle condition that was already established in the proof of Theorem 11.
In summary, $\mathfrak{k}$ is a indeed a Lie superalgebra with respect to the brackets we have chosen above provided every Killing spinor has κ and ϑ in (179) obeying all the conditions in (180). Let us now return to resolve these conditions. For simplicity, we shall assume henceforth that the connection C is flat (i.e. $G = 0$).
The condition $\mathscr{L}_\kappa g = 0$ in (180) follows immediately from (184) (since (184) implies $\nabla\kappa^\flat = \frac{1}{2}d\kappa^\flat$).
The condition $[\varphi_\mu, \vartheta] = 0$ in (180) implies that, at each point in M, either $\vartheta = 0$ or else φ_μ must be proportional to ϑ. Having assumed that C is flat, if the condition $\hat{\nabla}_\mu\vartheta = 0$ in (180) is satisfied, then we can always fix a gauge (i.e. for an appropriate local $Sp(1)$ transformation) in which ϑ is constant. So either ϑ is identically zero on M or else $\varphi_\mu = \psi_\mu\vartheta$, for some $\psi \in \Omega^1(M)$.
To make further progress, we now require the identity

$$\begin{aligned}\hat{\nabla}_\mu \omega^{AB}_{\nu\rho\sigma} = {} & 6H_{\mu[\nu}{}^\tau \omega^{AB}_{\rho\sigma]\tau} - 6\varphi_\mu^{(A}{}_C\omega^{B)C}_{\nu\rho\sigma} - 6\varphi^{(A}_{[\nu}{}_C\omega^{B)C}_{\rho\sigma]\mu} \\ & + 6g_{\mu[\nu}(\varphi^{AB}_\rho \kappa_{\sigma]} + \varphi^{\tau(A}{}_C\omega^{B)C}_{\rho\sigma]\tau}) - \varepsilon_{\mu\nu\rho\sigma\tau\theta}\varphi^{\tau AB}\kappa^\theta ,\end{aligned} \tag{186}$$

which can be derived (with some effort) from the definition of ω below (179) using the Killing spinor equation (157). Parentheses around indices denote symmetrisation while brackets denote skew-symmetrisation (with weight one in both cases). Skew-symmetrising $[\mu\nu\rho\sigma]$ in (186) gives a useful subsidiary identity

$$\hat{\nabla}_{[\mu}\omega^{AB}_{\nu\rho\sigma]} = 6H^-_{[\mu\nu}{}^\tau\omega^{AB}_{\rho\sigma]\tau} - \varepsilon_{\mu\nu\rho\sigma\tau\theta}\varphi^{\tau AB}\kappa^\theta , \tag{187}$$

where $H^\pm = \frac{1}{2}(H \pm \tilde{H}) \in \Omega^3_\pm(M)$ denotes the self-dual and anti-self-dual projections of H. The fact that only H^- appears in (187) is due to the identity

$$X^\pm_{[\mu\nu}{}^\tau Y^\pm_{\rho\sigma]\tau} = 0 , \tag{188}$$

which holds for any $X^\pm, Y^\pm \in \Omega^3_\pm(M)$ with the same chirality (c.f. Lemma 15).
The identity (187) defines $d^{\hat{\nabla}}\omega$. Having assumed that $G = 0$, acting with $\star d^{\hat{\nabla}}$ on $d^{\hat{\nabla}}\omega$ must give zero. Using (187) together with (186) and (184) to evaluate this operation yields (after some simplification) the

following expression for the (gauged) Lie derivative of φ along κ:

$$\begin{aligned}\hat{\mathscr{L}}_\kappa \varphi^{AB}_\mu &= \kappa_\mu \hat{\nabla}^\nu \varphi^{AB}_\nu + 2H^-_{\mu\nu\rho}\varphi^{\nu\, AB}\kappa^\rho + \varphi_\mu^{(A}{}_C\vartheta^{B)C} + H^{-\,\nu\rho\sigma}\varphi_\sigma^{(A}{}_C\omega^{B)C}_{\mu\nu\rho} \\ &\quad + \tfrac{1}{6}(\nabla_\mu H^-_{\nu\rho\sigma} - 6H^+_{\mu[\nu}{}^\tau H^-_{\rho\sigma]\tau})\omega^{AB\,\nu\rho\sigma} - g^{\sigma\tau}(\nabla_\tau H^-_{\nu\rho\sigma} - 6H^+_{\tau[\nu}{}^\theta H^-_{\rho\sigma]\theta})\omega^{AB\,\nu\rho}_\mu\ .\end{aligned} \tag{189}$$

Notice that the term $\varphi_\mu^{(A}{}_C\vartheta^{B)C}$ in the first line vanishes as a consequence of the condition $[\varphi,\vartheta]=0$ in (180). The remaining terms in (189) would vanish identically if φ and H obey

$$\hat{\nabla}^\nu \varphi^{AB}_\nu = 0\ , \quad H^-_{\mu\nu\rho}\varphi^{\rho\, AB} = 0\ , \quad \nabla_\mu H^-_{\nu\rho\sigma} - 6H^+_{\mu[\nu}{}^\tau H^-_{\rho\sigma]\tau} = 0\ . \tag{190}$$

To articulate the third condition in (190) more easily, let us define the connection $\nabla^+_X Y = \nabla_X Y + 2h^+(X,Y)$, with skew-symmetric torsion defined by $g(h^+(X,Y),Z) = H^+(X,Y,Z)$, for all $X,Y,Z \in \mathfrak{X}(M)$. The third condition in (190) just says that $\nabla^+ H^- = 0$.

The action of $\hat{\nabla}$ on ϑ can be evaluated using the definition (179) together with the identity (186). After some simplification, and applying the condition $[\varphi,\vartheta]=0$, this gives

$$\hat{\nabla}_\mu \vartheta^{AB} = (\nabla_\mu H^-_{\nu\rho\sigma} - 6H^+_{\mu[\nu}{}^\tau H^-_{\rho\sigma]\tau})\omega^{AB\,\nu\rho\sigma} + 12H^-_{\mu\nu\rho}\varphi^{\nu\,AB}\kappa^\rho - 12H^{-\,\nu\rho\sigma}\varphi_\sigma^{(A}{}_C\omega^{B)C}_{\mu\nu\rho}\ . \tag{191}$$

So $\hat{\nabla}\vartheta = 0$ if φ and H obey the second two conditions in (190).

To summarise, thus far we have shown that the three conditions on φ and H in (190), together with $[\varphi,\vartheta]=0$ and $G=0$, are sufficient to guarantee that all the conditions except $\mathscr{L}_\kappa H = 0$ in (180) are satisfied.

Taking the exterior derivative of the exact two-form defined by (184) (i.e. $d^2\kappa^\flat = 0$) and using the identity (186) provides us with the following expression for the Lie derivative of H along κ:

$$\mathscr{L}_\kappa H_{\mu\nu\rho} = -4\kappa^\sigma\nabla_{[\mu}H_{\nu\rho\sigma]} + 3\omega^{AB\,\sigma}_{[\mu\nu}(\hat{\nabla}_{\rho]}\varphi_\sigma + 2[\varphi_{\rho]},\varphi_\sigma] + 2H_{\rho]\sigma\tau}\varphi^\tau)_{AB} + 24H^-_{[\mu\nu}{}^\tau\omega^{AB}_{\rho\sigma]\tau}\varphi^\sigma_{AB}\ . \tag{192}$$

Even if we demand that H is closed (so that the first term in (192) vanishes identically), in general the three conditions on φ and H in (190) will not be sufficient to guarantee that $\mathscr{L}_\kappa H = 0$. In order to proceed, we will now consider two special cases that yield distinct branches of solutions of all the conditions in (180).

The first branch is defined by taking

$$d^{\hat{\nabla}}\star\varphi = 0\ , \quad H^- = 0\ , \quad dH = 0\ . \tag{193}$$

The first two conditions above ensure (190) are satisfied and therefore all the conditions except $\mathscr{L}_\kappa H = 0$ in (180) are guaranteed. Notice that H being self-dual implies that $\vartheta = 0$ identically (i.e. $[\mathfrak{k}_{\bar 1},\mathfrak{k}_{\bar 1}] \subset \mathfrak{K}$). From (192), we see that

$$\mathscr{L}_\kappa H_{\mu\nu\rho} = 3\omega^{AB\,\sigma}_{[\mu\nu}(\hat{\nabla}_{\rho]}\varphi_\sigma + 2[\varphi_{\rho]},\varphi_\sigma] + 2H_{\rho]\sigma\tau}\varphi^\tau)_{AB}\ , \tag{194}$$

and it is not obvious that the right hand side is zero. To prove that this is in fact the case, notice that $\mathscr{L}_\kappa\kappa = [\kappa,\kappa] = 0$ and $\hat{\mathscr{L}}_\kappa\omega = (\iota_\kappa d^{\hat{\nabla}} + d^{\hat{\nabla}}\iota_\kappa)\omega = 0$ (using $\iota_\kappa d^{\hat{\nabla}}\omega = 0$ from (187) and $\iota_\kappa\omega = 0$ from Lemma 6). Therefore, because we have already ensured that $\hat{\mathscr{L}}_\kappa\varphi = 0$, using (184) to evaluate $\mathscr{L}_\kappa\nabla\kappa^\flat$ and (186) to evaluate $\hat{\mathscr{L}}_\kappa\hat{\nabla}\omega$, we deduce that

$$\kappa^\mu\mathscr{L}_\kappa H_{\mu\nu\rho} = 0\ , \quad \mathscr{L}_\kappa H_{\mu[\nu}{}^\tau\omega^{AB}_{\rho\sigma]\tau} = 0\ . \tag{195}$$

But since $\mathscr{L}_\kappa H$ and ω^{AB} are self-dual three-forms, the identity (188) implies that the second condition above is equivalent to

$$\omega^{AB\,\tau}_{\mu[\nu}\mathscr{L}_\kappa H_{\rho\sigma]\tau} = 0\ . \tag{196}$$

So we have self-dual three-forms $\mathscr{L}_\kappa H$ and ω^{AB} obeying (196), with $\iota_\kappa\mathscr{L}_\kappa H = 0$ (from (195)) and $\iota_\kappa\omega^{AB} = 0$ (from Lemma 6). Furthermore, the Killing spinor bilinears κ and ω^{AB} are nowhere vanishing and κ is everywhere null. From Lemma 17 (identifying $\alpha = \mathscr{L}_\kappa H$, $\beta = \omega^{AB}$ and $N = \kappa$), it therefore follows that

$\mathscr{L}_\kappa H$ must equal some locally defined function multiplying ω^{AB}, for any choice of A and B. In particular, we must have

$$\mathscr{L}_\kappa H = f_1\omega^{11} = f_2\omega^{12} = f_3\omega^{22} \,, \tag{197}$$

for some locally defined functions f_1, f_2 and f_3. However, at any point in M, we know from Lemma 5 that ω^{11}, ω^{12} and ω^{22} are linearly independent. Therefore (197) can only be true if $\mathscr{L}_\kappa H = 0$, as required. So the conditions (193) for this branch do indeed imply (180) and therefore guarantee the existence of a Killing superalgebra. Since $\vartheta = 0$ here, it is possible to define a Killing superalgebra $\mathfrak{k}$ with $\mathfrak{k}_{\bar{0}} = \mathfrak{K}$, i.e. ignoring $\mathfrak{R}$ completely. The Killing superalgebras in this case are therefore naturally associated with the Spencer cohomology calculation that led to Theorem 10, where we ignored the R-symmetry.

The second branch is defined by taking

$$\varphi = 0 \,, \quad \nabla^+ H^- = 0 \,, \quad dH = 0 \,. \tag{198}$$

The first two conditions above ensure that (190) are satisfied while the third condition ensures that $\mathscr{L}_\kappa H = 0$. Thus, all the conditions in (180) are satisfied and the existence of a Killing superalgebra is guaranteed. Of course, if $H^- \neq 0$, we could have $\vartheta \neq 0$ for one or more Killing spinors on this branch, in which case $\mathfrak{R}$ must be included in the Killing superalgebra, just as we would expect from Theorem 11.

8. Maximally supersymmetric backgrounds

We now investigate which geometries admit the "maximal number of Killing spinors"; that is, for which the dimension of the space of solutions to equation (96) is maximal. This is equivalent to demanding that the curvature of the connection

$$\mathscr{D}_X = \nabla_X - \iota_X H + X\cdot\varphi + 2\varphi\cdot X \tag{199}$$

vanishes. As usual φ is a one-form with values in the Lie algebra of the R-symmetry and H a 3-form. We will not take H to be self-dual at this moment, but will comment on any simplifications which result from that assumption.

8.1. The curvature of the superconnection.

Let us write the connection as $\mathscr{D} = \nabla - \beta$, where

$$(\beta_\mu\varepsilon)^A = \tfrac12 H_{\mu\rho\sigma}\Gamma^{\rho\sigma}\varepsilon^A - 3\varphi_\mu{}^A{}_B\varepsilon^B + \varphi^{\sigma A}{}_B\Gamma_{\mu\sigma}\varepsilon^B. \tag{200}$$

The curvature $\mathscr{R}$ of $\mathscr{D}$ is given by

$$\mathscr{R}_{\mu\nu}{}^A{}_B = \tfrac14 R_{\mu\nu\alpha\beta}\Gamma^{\alpha\beta}\delta^A{}_B + \nabla_\mu\beta_\nu{}^A{}_B - \nabla_\nu\beta_\mu{}^A{}_B - [\beta_\mu,\beta_\nu]^A{}_B, \tag{201}$$

which can be decomposed into components

$$\mathscr{R}_{\mu\nu}{}^A{}_B = \tfrac12 \mathscr{T}_{\mu\nu\alpha\beta}\Gamma^{\alpha\beta}\delta^A{}_B + \mathscr{U}_{\mu\nu}{}^A{}_B + \tfrac12\mathscr{V}_{\mu\nu\alpha\beta}{}^A{}_B\Gamma^{\alpha\beta}. \tag{202}$$

The tensor $\mathscr{T}$ is skew-symmetric in the first and last pairs of indices, so it is a section through a bundle associated to the representation $\Lambda^2 V\otimes\Lambda^2 V$ of $\mathfrak{so}(V)$. Similarly $\mathscr{U}$ (resp. $\mathscr{V}$) is a section through a bundle associated to the representation $\Lambda^2 V\otimes\mathfrak{sp}(1)$ (resp. $\Lambda^2 V\otimes\Lambda^2 V\otimes\mathfrak{sp}(1)$) of $\mathfrak{so}(V)\oplus\mathfrak{sp}(1)$.

The explicit expressions of the components are

$$\begin{aligned}\mathscr{T}_{\mu\nu\alpha\beta} = \tfrac12 R_{\mu\nu\alpha\beta} &+ \nabla_\mu H_{\nu\alpha\beta} - \nabla_\nu H_{\mu\alpha\beta} + 2\left(\langle H_{\mu\alpha}, H_{\nu\beta}\rangle - \langle H_{\mu\beta}, H_{\nu\alpha}\rangle\right)\\ &+ \left(g_{\mu\alpha}\langle\varphi_\beta,\varphi_\nu\rangle - g_{\mu\beta}\langle\varphi_\alpha,\varphi_\nu\rangle - g_{\nu\alpha}\langle\varphi_\beta,\varphi_\mu\rangle + g_{\nu\beta}\langle\varphi_\alpha,\varphi_\mu\rangle\right)\\ &- \left\langle\varphi^\lambda,\varphi_\lambda\right\rangle\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right),\end{aligned} \tag{203}$$

with $\langle H_{\mu\nu}, H_{\alpha\beta}\rangle := H_{\mu\nu}{}^\lambda H_{\alpha\beta\lambda}$ and $\langle\varphi_\mu,\varphi_\nu\rangle = \varphi_\mu{}^{AB}\varphi_{\nu\,AB}$, and, omitting the R-symmetry indices,

$$\mathscr{U}_{\mu\nu} = -3\left(\nabla_\mu\varphi_\nu - \nabla_\nu\varphi_\mu\right) - 8[\varphi_\mu,\varphi_\nu], \tag{204}$$

and

$$\begin{aligned}\mathscr{V}_{\mu\nu\alpha\beta} = \nabla_\mu\varphi_\beta g_{\nu\alpha} - \nabla_\mu\varphi_\alpha g_{\nu\beta} - \nabla_\nu\varphi_\beta g_{\mu\alpha} + \nabla_\nu\varphi_\alpha g_{\mu\beta} + 4\,(H_{\mu\nu\alpha}\varphi_\beta - H_{\mu\nu\beta}\varphi_\alpha) \\ - 2\,(H_{\mu\beta\lambda}g_{\nu\alpha} - H_{\mu\alpha\lambda}g_{\nu\beta} - H_{\nu\beta\lambda}g_{\mu\alpha} + H_{\nu\alpha\lambda}g_{\mu\beta})\,\varphi^\lambda - \epsilon_{\mu\nu\alpha\beta\rho\sigma}[\varphi^\rho, \varphi^\sigma] \\ + 3\,([\varphi_\mu, \varphi_\beta]g_{\nu\alpha} - [\varphi_\mu, \varphi_\alpha]g_{\nu\beta} - [\varphi_\nu, \varphi_\beta]g_{\mu\alpha} + [\varphi_\nu, \varphi_\alpha]g_{\mu\beta})\,. \end{aligned} \tag{205}$$

8.2. **Zero curvature conditions.** We begin our analysis of the zero curvature conditions. Let us start with the condition $\mathscr{T}_{\mu\nu\alpha\beta} = 0$. The tensor $\mathscr{T}$ is a section through a bundle associated to $\Lambda^2 V \otimes \Lambda^2 V$ and this representation decomposes as follows into irreducible components under $\mathfrak{so}(V)$:

$$\begin{aligned}\Lambda^2 V \otimes \Lambda^2 V &= \odot^2(\Lambda^2 V) \oplus \Lambda^2(\Lambda^2 V) \\ &= \left(\Lambda^4 V \oplus \Lambda^0 V \oplus \odot^2_0 V \oplus W\right) \oplus \left(\Lambda^2 V \oplus (V \otimes \Lambda^3_+ V)_0 \oplus (V \otimes \Lambda^3_- V)_0\right),\end{aligned} \tag{206}$$

where $(V \otimes \Lambda^3_\pm V)_0$ is the kernel of the natural contraction $V \otimes \Lambda^3_\pm V \to \Lambda^2 V$ and W is the module of Weyl curvature tensors.

Setting the $\Lambda^2(\Lambda^2 V)$ component to zero, we find

$$\begin{aligned}0 &= \mathscr{T}_{\mu\nu\alpha\beta} - \mathscr{T}_{\alpha\beta\mu\nu} \\ &= \nabla_\mu H_{\nu\alpha\beta} - \nabla_\nu H_{\mu\alpha\beta} - \nabla_\alpha H_{\beta\mu\nu} + \nabla_\beta H_{\alpha\mu\nu},\end{aligned} \tag{207}$$

which we can rewrite as

$$\nabla_{[\mu} H_{\nu]\alpha\beta} = \nabla_{[\alpha} H_{\beta]\mu\nu}. \tag{208}$$

Skew-symmetrising in the last three indices we find

$$\begin{aligned}0 &= \mathscr{T}_{\mu[\nu\alpha\beta]} \\ &= \nabla_\mu H_{\nu\alpha\beta} - \tfrac{1}{3}\,(\nabla_\nu H_{\alpha\beta\mu} + \nabla_\alpha H_{\beta\nu\mu} + \nabla_\beta H_{\nu\alpha\mu}) + \tfrac{4}{3}\,(\langle H_{\mu\alpha}, H_{\nu\beta}\rangle - \langle H_{\mu\beta}, H_{\nu\alpha}\rangle - \langle H_{\mu\nu}, H_{\alpha\beta}\rangle)\,,\end{aligned} \tag{209}$$

whereas completely skew-symmetrising gives

$$\begin{aligned}0 &= \mathscr{T}_{[\mu\nu\alpha\beta]} \\ &= \nabla_{[\mu} H_{\nu]\alpha\beta} + \nabla_{[\alpha} H_{\beta]\mu\nu} + \tfrac{4}{3}\,(\langle H_{\mu\alpha}, H_{\nu\beta}\rangle - \langle H_{\mu\beta}, H_{\nu\alpha}\rangle - \langle H_{\mu\nu}, H_{\alpha\beta}\rangle)\,.\end{aligned} \tag{210}$$

Comparing the two equations (209) and (210) we see that

$$\nabla_{[\mu} H_{\nu]\alpha\beta} + \nabla_{[\alpha} H_{\beta]\mu\nu} = 2\nabla_\mu H_{\nu\alpha\beta}, \tag{211}$$

which together with equation (208) gives

$$\nabla_\mu H_{\nu\alpha\beta} + \nabla_\nu H_{\mu\alpha\beta} = 0, \tag{212}$$

which says that the covariant derivative of H is a 4-form. In other words $\nabla H = \frac{1}{4} dH$ and H is a coclosed conformal Killing 3-form, or a Killing 3-form in the nomenclature of [33].

The totally skew-symmetric component then finally gives

$$\nabla_\mu H_{\nu\alpha\beta} = \tfrac{2}{3}\,(\langle H_{\mu\nu}, H_{\alpha\beta}\rangle - \langle H_{\mu\alpha}, H_{\nu\beta}\rangle + \langle H_{\mu\beta}, H_{\nu\alpha}\rangle)\,. \tag{213}$$

The algebraic curvature tensor components $(\Lambda^0 V \oplus \odot^2_0 V \oplus W)$ of $\mathscr{T}$ give the Riemann tensor in terms of H and φ.

The vanishing of the $\mathscr{U}$-component of the curvature gives the equation

$$0 = \tfrac{1}{2}\mathscr{U}_{\mu\nu} = -3\nabla_{[\mu}\varphi_{\nu]} - 4[\varphi_\mu, \varphi_\nu]. \tag{214}$$

Next let us consider the vanishing of the $\mathscr{V}$-component of the curvature. Totally skew-symmetrising and omitting the R-symmetry indices, we obtain

$$0 = \mathscr{V}_{[\mu\nu\alpha\beta]} = 8H_{[\mu\nu\alpha}\varphi_{\beta]} - \epsilon_{\mu\nu\alpha\beta\rho\sigma}[\varphi^\rho, \varphi^\sigma], \tag{215}$$

which can be rewritten as

$$[\varphi_\mu, \varphi_\nu] = \widetilde{H}_{\mu\nu\lambda}\varphi^\lambda, \tag{216}$$

with $\widetilde{H}$ the Hodge dual of H.
The component along $\Lambda^2(\Lambda^2 V)$ gives

$$\begin{aligned}
0 &= \tfrac{1}{2}(\mathscr{V}_{\mu\nu\alpha\beta} - \mathscr{V}_{\alpha\beta\mu\nu}) \\
&= \nabla_{[\mu}\varphi_{\beta]}g_{\nu\alpha} - \nabla_{[\mu}\varphi_{\alpha]}g_{\nu\beta} - \nabla_{[\nu}\varphi_{\beta]}g_{\mu\alpha} + \nabla_{[\nu}\varphi_{\alpha]}g_{\mu\beta} \\
&\quad + 2\,(H_{\mu\nu\alpha}\varphi_\beta - H_{\mu\nu\beta}\varphi_\alpha - H_{\alpha\beta\mu}\varphi_\nu + H_{\alpha\beta\nu}\varphi_\mu) \\
&\quad - 2\,(H_{\mu\beta\lambda}g_{\nu\alpha} - H_{\mu\alpha\lambda}g_{\nu\beta} - H_{\nu\beta\lambda}g_{\mu\alpha} + H_{\nu\alpha\lambda}g_{\mu\beta})\,\varphi^\lambda \\
&\quad + 3\,([\varphi_\mu,\varphi_\beta]g_{\nu\alpha} - [\varphi_\mu,\varphi_\alpha]g_{\nu\beta} - [\varphi_\nu,\varphi_\beta]g_{\mu\alpha} + [\varphi_\nu,\varphi_\alpha]g_{\mu\beta})\,.
\end{aligned} \tag{217}$$

Using equations (214) and (216), we may rewrite this equation as

$$\begin{aligned}
0 &= 2\,(H_{\mu\nu\alpha}\varphi_\beta - H_{\mu\nu\beta}\varphi_\alpha - H_{\alpha\beta\mu}\varphi_\nu + H_{\alpha\beta\nu}\varphi_\mu) \\
&\quad - 2\,(H_{\mu\beta\lambda}g_{\nu\alpha} - H_{\mu\alpha\lambda}g_{\nu\beta} - H_{\nu\beta\lambda}g_{\mu\alpha} + H_{\nu\alpha\lambda}g_{\mu\beta})\,\varphi^\lambda \\
&\quad + \tfrac{5}{3}\left(\widetilde{H}_{\mu\beta\lambda}g_{\nu\alpha} - \widetilde{H}_{\mu\alpha\lambda}g_{\nu\beta} - \widetilde{H}_{\nu\beta\lambda}g_{\mu\alpha} + \widetilde{H}_{\nu\alpha\lambda}g_{\mu\beta}\right)\varphi^\lambda.
\end{aligned} \tag{218}$$

Contracting with $g^{\nu\alpha}$, we find

$$\tfrac{5}{3}\widetilde{H}_{\mu\beta\lambda}\varphi^\lambda = H_{\mu\beta\lambda}\varphi^\lambda, \tag{219}$$

and reinserting this into equation (218), we arrive at

$$2\,(H_{\mu\nu\alpha}\varphi_\beta - H_{\mu\nu\beta}\varphi_\alpha - H_{\alpha\beta\mu}\varphi_\nu + H_{\alpha\beta\nu}\varphi_\mu) = (H_{\mu\beta\lambda}g_{\nu\alpha} - H_{\mu\alpha\lambda}g_{\nu\beta} - H_{\nu\beta\lambda}g_{\mu\alpha} + H_{\nu\alpha\lambda}g_{\mu\beta})\,\varphi^\lambda. \tag{220}$$

Lemma 25. *Equation* (220) *is equivalent to* $H_{\mu\nu\alpha}\varphi_\beta = 0$.

Proof. Let $p \in M$ and suppose that $\varphi|(p) \neq 0$, so that some component φ^{AB} is different from zero at p. We will let $a = \varphi^{AB}(p)$ and show that $H(p) = 0$.
Equation (220) for the component a becomes

$$2\,(H_{\mu\nu\alpha}a_\beta - H_{\mu\nu\beta}a_\alpha - H_{\alpha\beta\mu}a_\nu + H_{\alpha\beta\nu}a_\mu) = (H_{\mu\beta\lambda}g_{\nu\alpha} - H_{\mu\alpha\lambda}g_{\nu\beta} - H_{\nu\beta\lambda}g_{\mu\alpha} + H_{\nu\alpha\lambda}g_{\mu\beta})\,a^\lambda. \tag{221}$$

Skew-symmetrising in $[\mu\nu\alpha]$, we find

$$H_{\mu\nu\alpha}a_\beta + H_{\beta[\mu\nu}a_{\alpha]} = g_{\beta[\mu}H_{\nu\alpha]\lambda}a^\lambda. \tag{222}$$

Contracting equation (221) with a^β, we find that

$$H_{\mu\nu\alpha}a^2 = 0, \tag{223}$$

whereas contracting equation (222) with a^μ and using equation (223), we arrive at

$$a_\beta H_{\nu\alpha\lambda}a^\lambda = a_{[\nu}H_{\alpha]\beta\lambda}a^\lambda. \tag{224}$$

Let's multiply this equation by a_μ and skew-symmetrise in $[\mu\nu\alpha]$ to obtain

$$a_\beta a_{[\mu}H_{\nu\alpha]\lambda}a^\lambda = 0, \tag{225}$$

which, since $a \neq 0$, is equivalent to

$$a_{[\mu}H_{\nu\alpha]\lambda}a^\lambda = 0. \tag{226}$$

Adding equations (224) and (226), we arrive at

$$H_{\nu\alpha\lambda}a^\lambda = 0. \tag{227}$$

Inserting this into equation (222), we get

$$H_{\mu\nu\alpha}a_\beta + H_{\beta[\mu\nu}a_{\alpha]} = 0, \tag{228}$$

and into equation (221),

$$H_{\mu\nu\alpha}a_\beta - H_{\mu\nu\beta}a_\alpha - H_{\alpha\beta\mu}a_\nu + H_{\alpha\beta\nu}a_\mu = 0. \tag{229}$$

Subtracting the two equations, we find

$$H_{\mu\nu(\alpha}a_{\beta)} = 0. \tag{230}$$

Now let's contract this equation with b^β, where b is a vector such that $b^\mu a_\mu = 1$, to find

$$H_{\mu\nu\alpha} + H_{\mu\nu\beta} a_\alpha b^\beta = 0. \tag{231}$$

Contracting this equation with b^α now, we find $H_{\mu\nu\alpha} b^\alpha = 0$, which inserted in the previous equation, gives $H_{\mu\nu\alpha} = 0$, as desired. □

We have proved most of the following.

Proposition 26. *Let (M, g, H, φ) be a (connected) lorentzian six-dimensional spin manifold endowed with a 3-form H and a 1-form φ with values in $\mathfrak{sp}(1)$. Assume (M, g, H, φ) is maximally supersymmetric, that is, the dimension of the space of solutions to equation* (96) *is maximal. Then either (1) $H = 0$ or (2) $\varphi = 0$. Moreover:*

(1) In the first case $\varphi = a \otimes R$ is decomposable, where $a \in \Omega^1(M)$ is a parallel 1-form on M and $R \in \mathfrak{sp}(1)$ a fixed element of the R-symmetry algebra;

(2) In the second case H is a coclosed conformal Killing 3-form (i.e., $\nabla H = \frac{1}{4} dH$) with covariant derivative given by

$$\nabla_\mu H_{\nu\alpha\beta} = \tfrac{2}{3}\left(\langle H_{\mu\nu}, H_{\alpha\beta}\rangle - \langle H_{\mu\alpha}, H_{\nu\beta}\rangle + \langle H_{\mu\beta}, H_{\nu\alpha}\rangle\right). \tag{232}$$

In both cases, the algebraic curvature tensor components ($\Lambda^0 V \oplus \odot^2_0 V \oplus W$) of the tensor (203) *give the Riemann curvature tensor in terms of φ and, respectively, H.*

Proof. Lemma 25 shows that any point $p \in M$, either $\varphi = 0$ or $H = 0$ (or possibly both). In particular, this means that at all points, $H_{\mu\nu\lambda}\varphi^\lambda = 0$, so that also $\widetilde{H}_{\mu\nu\lambda}\varphi^\lambda = 0$, $[\varphi_\mu, \varphi_\nu] = 0$ and $\nabla_{[\mu}\varphi_{\nu]} = 0$.
The component of $\mathscr{V}$ along $\odot^2(\Lambda^2 V)$ gives, after using equation (215),

$$\nabla_{(\mu}\varphi_{\beta)} g_{\nu\alpha} - \nabla_{(\mu}\varphi_{\alpha)} g_{\nu\beta} - \nabla_{(\nu}\varphi_{\beta)} g_{\mu\alpha} + \nabla_{(\nu}\varphi_{\alpha)} g_{\mu\beta} = 0, \tag{233}$$

which upon contraction with $g^{\nu\alpha}$ gives

$$4\nabla_{(\mu}\varphi_{\beta)} + \nabla^\lambda \varphi_\lambda g_{\mu\beta} = 0. \tag{234}$$

Contracting with $g^{\mu\beta}$ we find $\nabla^\lambda\varphi_\lambda = 0$ and inserting back into the previous equation,

$$\nabla_{(\mu}\varphi_{\beta)} = 0. \tag{235}$$

Together with $\nabla_{[\mu}\varphi_{\nu]} = 0$, we conclude that φ is parallel. Parallel sections of vector bundles are determined by their value at any given point, hence if $\varphi = 0$ at any point, it is identically zero. In other words, either φ is identically zero or, by Lemma 25, H is identically zero. (Of course, it is possible that both are identically zero, which corresponds to the trivial (flat) background.)
In the first case $\varphi = a \otimes R$ for some parallel $a \in \Omega^1(M)$ and constant $R \in \mathfrak{sp}(1)$, since $\varphi : TM \to \mathfrak{sp}(1)$ has a 1-dimensional range at any point $p \in M$ (due to $[\varphi_\mu, \varphi_\nu] = 0$) and it is parallel. The rest is clear. □

In summary, we have two branches of nontrivial backgrounds, which we will analyse in turn.

8.3. **First branch: $H = 0$.** Here $\varphi \neq 0$ and $\varphi_\mu{}^A{}_B = a_\mu R^A{}_B$, where R is a fixed element of the R-symmetry algebra $\mathfrak{sp}(1)$ and the one-form a is parallel. Without loss of generality we can normalise R so that $R^{AB}R_{AB} = 1$ or, equivalently, that $\mathrm{tr}(R^2) = -1$. The fact that the 1-form a is parallel can be seen also from the vanishing of the $\mathscr{T}$-component (203) of the curvature of the spinor connection, which in this branch becomes

$$R_{\mu\nu\alpha\beta} = 2a^2(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) - 2(g_{\mu\alpha}a_\beta a_\nu - g_{\mu\beta}a_\alpha a_\nu - g_{\nu\alpha}a_\beta a_\mu + g_{\nu\beta}a_\alpha a_\mu), \tag{236}$$

and noticing that $R_{\mu\nu\alpha\beta}a^\beta = 0$.
The causal type of a parallel vector is constant, so we may distinguish between three cases depending on whether a is null, spacelike or timelike. The discussion breaks up naturally into two cases, depending on whether or not the squared norm a^2 of a vanishes. In all cases, it follows from the expression (236) of the Riemann tensor that the Weyl tensor vanishes and hence that all geometries are conformally flat.

8.3.1. $a^2 \neq 0$. Since a is parallel, nowhere-vanishing and $a^2 \neq 0$, the de Rham decomposition theorem says that (M, g) is locally isometric to a product: $M = N \times \mathbb{R}$, where $\mathbb{R}$ is either timelike or spacelike according to the causal type of a. To understand the geometry of N, we rewrite the Riemann tensor in equation (236) as follows:

$$R_{\mu\nu\alpha\beta} = 2a^2 \left(h_{\mu\alpha}h_{\nu\beta} - h_{\mu\beta}h_{\nu\alpha}\right), \tag{237}$$

where the tensor $h_{\mu\nu} := g_{\mu\nu} - \frac{a_\mu a_\nu}{a^2}$. Note that h coincides with the induced metric on the distribution $a^\perp$ perpendicular to a, which is the tangent bundle of N. The above form of the Riemann tensor makes it evident that N has constant sectional curvature, with Ricci tensor

$$R_{\mu\beta} := R_\mu{}^\nu{}_{\nu\beta} = -8a^2 h_{\mu\beta}. \tag{238}$$

Therefore if a is timelike, so that $a^2 < 0$, (M, g) is locally isometric to $(\mathbb{R}, -dt^2) \times S^5$, where S^5 is a round 5-sphere with scalar curvature $-40a^2$; whereas if a is spacelike, (M, g) is locally isometric to $(\mathbb{R}, dt^2) \times \mathrm{AdS}_5$, with AdS_5 the anti-de Sitter spacetime with scalar curvature $-40a^2$.

8.3.2. a *is null.* If a is non-zero and null, (M, g) is a Brinkmann space, with Riemann curvature tensor given by

$$R_{\mu\nu\alpha\beta} = -2(g_{\mu\alpha}a_\beta a_\nu - g_{\mu\beta}a_\alpha a_\nu - g_{\nu\alpha}a_\beta a_\mu + g_{\nu\beta}a_\alpha a_\mu). \tag{239}$$

It is clear by inspection of the above expression for the Riemann curvature tensor, that the metric is both conformally flat and scalar flat. Furthermore, $R(a^\perp, a^\perp) = 0$, so that the transverse geometry is flat and since a and g are both parallel, so is the Riemann tensor. Hence (M, g) is locally isometric to a (possibly decomposable) Cahen–Wallach plane wave [34], with metric

$$g = 2dx^+dx^- + \sum_{i,j=1}^{4} B_{ij}x^i x^j (dx^-)^2 + \sum_{i=1}^{4} (dx^i)^2, \tag{240}$$

where the parallel null vector is $a = \partial_+$. The only nonzero components of the Weyl tensor of the metric g are

$$W_{-ij-} = B_{ij} - \tfrac{1}{4}(\operatorname{tr} B)\delta_{ij}, \tag{241}$$

so that g is conformally flat if and only if B is a scalar matrix. From the explicit form of the Riemann curvature tensor (239) we see that B is nonzero and up to a local diffeomorphism we may write the metric down as

$$g_\pm = 2dx^+dx^- \pm \tfrac{1}{4}\sum_{i=1}^{4} (x^i)^2 (dx^-)^2 + \sum_{i=1}^{4} (dx^i)^2. \tag{242}$$

Comparing the Riemann tensor of $g_\pm$ with (239) precisely selects the metric g_- (see also equation (13) in [28]). The metric in this background is (locally) isometric to the plane wave in [35]; but the Killing spinors here obey a different equation than those of $d = 6$ $(1, 0)$ supergravity. In other words, we are in the curious situation where the same geometry is maximally supersymmetric with respect to two different notions of Killing spinors.

8.4. **Second branch:** $\varphi = 0$. In the second branch, $\varphi = 0$ and the Killing spinors satisfy

$$\mathscr{D}_X\varepsilon = \nabla_X\varepsilon - \iota_X H \cdot \varepsilon = 0. \tag{243}$$

We may understand this equation as saying that Killing spinors are parallel with respect to (the spin lift of) the metric connection

$$D_X Y = \nabla_X Y + 2h(X, Y) \tag{244}$$

with skew-symmetric torsion $g(h(X,Y),Z) = H(X,Y,Z)$. Since the representation of $\mathfrak{so}(V)$ on Σ_+ is faithful, maximal supersymmetry exactly amounts to the flatness of D and, using Proposition 26, this condition is equivalent to

$$\begin{aligned}
\nabla_\mu H_{\nu\alpha\beta} &= \tfrac{2}{3}\left(\langle H_{\mu\nu}, H_{\alpha\beta}\rangle - \langle H_{\mu\alpha}, H_{\nu\beta}\rangle + \langle H_{\mu\beta}, H_{\nu\alpha}\rangle\right)\\
R_{\mu\nu\alpha\beta} &= -2\nabla_\mu H_{\nu\alpha\beta} + 2\nabla_\nu H_{\mu\alpha\beta} - 4\left(\langle H_{\mu\alpha}, H_{\nu\beta}\rangle - \langle H_{\mu\beta}, H_{\nu\alpha}\rangle\right)\\
&= -\tfrac{8}{3}\langle H_{\mu\nu}, H_{\alpha\beta}\rangle - \tfrac{4}{3}\langle H_{\mu\alpha}, H_{\nu\beta}\rangle + \tfrac{4}{3}\langle H_{\mu\beta}, H_{\nu\alpha}\rangle\,.
\end{aligned} \tag{245}$$

If the torsion 3-form H is, in addition, parallel with respect to ∇ (equivalently it is closed) then H obeys the Jacobi identity. More precisely, it is possible to use (245) to see that in this case (M,g) is locally isometric to a Lie group with a bi-invariant lorentzian metric, H is the Cartan 3-form of the group and D the parallelising connection (see [36, §2.3]). Furthermore H is also D-parallel. We recall that the general classification of lorentzian Lie groups is due to Medina [37].

Now, it is a deep result of Cahen and Parker that a simply connected, complete indecomposable *lorentzian* manifold (M,g) with a flat metric connection with skew-torsion H satisfies $\nabla H = 0$ automatically [38]. They also show that the assumption of indecomposability can be relaxed.

In summary, a maximally supersymmetric background (M,g,H) in the second branch of our classification is (up to local isometry) a Lie group with a bi-invariant lorentzian metric. The Lie algebra of such a Lie group is a six-dimensional Lie algebra with a lorentzian ad-invariant inner product and these have been listed in [28,36]. The corresponding backgrounds are:

(1) $\mathbb{R}^{5,1}$,
(2) $\mathbb{R}^{2,1}\times S^3$,
(3) $\mathbb{R}^3\times \mathrm{AdS}_3$,
(4) $\mathrm{AdS}_3\times S^3$,
(5) the plane wave (242) in [35].

We emphasise that the Cartan 3-form H may be chosen self-dual or antiself-dual in cases (1), (4) and (5) above. Solutions with self-dual Cartan 3-form correspond to the maximally supersymmetric backgrounds of $d=6$ $(1,0)$ supergravity.

We have proved the following classification result. We recall that in our conventions S is an irreducible representation of $\mathrm{Spin}(V)$ of quaternionic dimension 2.

Theorem 27. *Let (M,g) be a lorentzian six-dimensional spin manifold, with associated spinor bundle $\mathbb{S}\to M$ with typical fiber S. Let $H\in\Omega^3(M)$ be a 3-form and φ a 1-form on M with values in $\mathfrak{sp}(1)$. Let also*

$$\mathscr{D}_X\varepsilon := \nabla_X\varepsilon - \iota_X H\cdot\varepsilon + 3\varphi(X)\cdot\varepsilon - X\wedge\varphi\cdot\varepsilon\,, \tag{246}$$

be the associated spinor connection. If $\mathscr{D}$ is flat, then (M,g,H,φ) is locally isometric to one of the following:

(i) $\varphi = H = 0$ *and* (M,g) *Minkowski spacetime;*
(ii) $H=0$ *and* $\varphi = \alpha\otimes R$ *for some parallel 1-form* α *and fixed element* R *of the R-symmetry algebra* $\mathfrak{sp}(1)$ *with* $\mathrm{tr}(R^2)=-1$. *Depending on the causal type of* α:
  (a) (M,g) *the product* $(\mathbb{R},-dt^2)\times S^5$, *with* S^5 *the round 5-sphere of scalar curvature* $-40\alpha^2$, *for* $\alpha^2<0$;
  (b) (M,g) *the product* $(\mathbb{R},dt^2)\times\mathrm{AdS}_5$, *with* AdS_5 *the five-dimensional anti-de Sitter spacetime with scalar curvature* $-40\alpha^2$, *for* $\alpha^2>0$;
  (c) and if $\alpha^2=0$, *then* (M,g) *a conformally flat lorentzian symmetric plane wave with metric*

$$g = 2dx^+dx^- - \tfrac14\sum_{i=1}^4 (x^i)^2(dx^-)^2 + \sum_{i=1}^4 (dx^i)^2; \tag{247}$$

(iii) $\varphi=0$ *and* H *is the parallel Cartan 3-form of a six-dimensional Lie group with bi-invariant lorentzian metric. If* H *is in addition self-dual, these are the maximally supersymmetric backgrounds of* $d=6$ $(1,0)$ *supergravity.*

8.5. **Killing superalgebras and filtered deformations.** A natural question is whether the Killing spinors in the maximally supersymmetric backgrounds of Theorem 27 generate always a Lie superalgebra and if different backgrounds have different associated Killing superalgebras (in particular, if the plane wave that is maximally supersymmetric in two different senses has different Killing superalgebras for the different types of Killing spinors).

Concerning existence, there is clearly nothing to check for the trivial Minkowski background. In the first branch, where $H = 0$ and $\varphi = \alpha \otimes R$ is parallel (in particular coclosed), this follows from Theorem 20. In the second branch, the existence is guaranteed from the fact that $H = H^+ + H^-$ is closed and D-parallel, so that Theorem 24 applies. When H is self-dual, Theorem 20 is actually sufficient, giving rise to an ideal $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ of the general extended Killing superalgebra $\hat{\mathfrak{k}} = \hat{\mathfrak{k}}_{\bar{0}} \oplus \hat{\mathfrak{k}}_{\bar{1}}$.

The fact that maximally supersymmetric backgrounds are distinguished by their Killing superalgebras is a consequence of the general theory of filtered deformations (of subalgebras) of the Poincaré superalgebra, possibly extended by R-symmetries. The full line of arguments is as for the eleven-dimensional case [25, 26, 39] and four-dimensional case [19], and we will not replicate it here. We will review the main ingredients in the simpler case of "maximally supersymmetric" filtered deformations, which is enough for the purpose of this paper, and comment on any adjustment which results from the presence of R-symmetries.

Let

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2} \tag{248}$$

be a $\mathbb{Z}$-graded Lie subalgebra of the d=6 $(1,0)$ Poincaré superalgebra

$$\begin{aligned} \mathfrak{p} &= \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0 \\ &= V \oplus S \oplus \mathfrak{so}(V) , \end{aligned} \tag{249}$$

which satisfies $\mathfrak{a}_{-1} = S$ and $\mathfrak{a}_{-2} = V$. The fact that $\mathfrak{a}_{-1} = S$ means we have maximal supersymmetry, whereas $\mathfrak{a}_{-2} = V$ (which is forced by the local homogeneity theorem in [30]) means we are describing (locally) homogeneous geometries.

Definition 28. *A* filtered deformation *of $\mathfrak{a}$ is a Lie superalgebra $\mathfrak{g}$ supported on the same underlying vector space of $\mathfrak{a}$ whose Lie brackets have nonnegative total degree, with the zero-degree components coinciding with the Lie brackets of $\mathfrak{a}$.*

If we do not wish to mention the subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ explicitly, we simply say that $\mathfrak{g}$ is a (maximally supersymmetric) filtered subdeformation of $\mathfrak{p}$. The notion of a filtered subdeformation $\mathfrak{g}$ of the extended Poincaré superalgebra $\hat{\mathfrak{p}}$ can be introduced in a completely analogous way. We note that the spin group $\mathrm{Spin}(V)$ naturally acts on $\mathfrak{p}$ by 0-degree Lie superalgebra automorphisms, so that any element $g \in \mathrm{Spin}(V)$ sends a graded subalgebra of $\mathfrak{p}$ into an (isomorphic) graded subalgebra of $\mathfrak{p}$ and filtered subdeformations into filtered subdeformations. A similar observation holds for the action of $\mathrm{Spin}(V) \times \mathrm{Sp}(1)$ on $\hat{\mathfrak{p}}$.

The $\mathbb{Z}_2$-grading of $\mathfrak{a}$ is compatible with the $\mathbb{Z}$-grading, in that $\mathfrak{a}_{\bar{0}} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-2}$ and $\mathfrak{a}_{\bar{1}} = \mathfrak{a}_{-1}$. In particular the components of the Lie brackets of a filtered subdeformation $\mathfrak{g}$ of $\mathfrak{p}$ or $\hat{\mathfrak{p}}$ have even (nonnegative) degree, which is at most four.

First, we wish to localise the Killing superalgebra associated to a maximally supersymmetric background (M, g, H, φ) of Theorem 27 at a point $p \in M$. The construction is parallel to that given in four-dimensions [19, §3.3] and eleven-dimensions [39, §3.1, §3.2].

First of all, elements of the Killing superalgebra may be identified with parallel sections of the supervector bundle $\mathscr{E} = \mathscr{E}_{\bar{0}} \oplus \mathscr{E}_{\bar{1}}$,

$$\mathscr{E}_{\bar{0}} = TM \oplus \mathfrak{so}(TM) \qquad \text{and} \qquad \mathscr{E}_{\bar{1}} = \mathbb{S} . \tag{250}$$

For extended Killing superalgebras, one needs to set $\mathscr{E}_{\bar{0}} = TM \oplus \mathfrak{so}(TM) \oplus \mathfrak{sp}(\mathcal{H})$. It is clear that Killing spinors and R-symmetries are parallel w.r.t. the connection $\mathscr{D}$, whereas it is a well-known fact that any Killing vector is identified with a parallel section of $TM \oplus \mathfrak{so}(TM)$ by the so-called Killing transport.

Hence, any element of the (extended) Killing superalgebra is determined by the value at $p \in M$ of the corresponding parallel section of $\mathscr{E}$ and the Killing superalgebra itself defines a graded subspace $\mathfrak{a}$ of the (extended) Poincaré superalgebra. It is not difficult to see that

$$\mathfrak{a} = \begin{cases} V \oplus S \oplus \mathfrak{h} & \text{if } H^- = 0\,, \\ V \oplus S \oplus (\mathfrak{h} \oplus \mathfrak{r}) & \text{if } H^- \neq 0\,, \end{cases} \tag{251}$$

for some subalgebra $\mathfrak{h}$ of $\mathfrak{so}(V)$. Tracking back the Lie algebra structure of the Killing superalgebra yields the following Lie brackets on $\mathfrak{a}$:

$$\begin{aligned}
[L, M] &= LM - ML \\
[L, A] &= 0 \\
[A, B] &= AB - BA \\
[L, s] &= \tfrac{1}{2}\omega_L \cdot s \\
[A, s] &= As \\
[L, v] &= Lv + \underbrace{[L, X_v] - X_{Lv}}_{\text{element of } \mathfrak{h}} \\
[A, v] &= 0 \\
[s, s] &= \kappa(s,s) + \underbrace{\gamma(s,s) - X_{\kappa(s,s)}}_{\text{element of } \mathfrak{h}} + \underbrace{\rho(s,s)}_{\text{element of } \mathfrak{r}} \\
[v, s] &= \underbrace{\beta_v s + \tfrac{1}{2}\omega_{X_v} \cdot s}_{\text{element of } S} \\
[v, w] &= \underbrace{X_v w - X_w v}_{\text{element of } V} + \underbrace{[X_v, X_w] + R(v,w) - X_{X_v w - X_w v}}_{\text{element of } \mathfrak{h}},
\end{aligned} \tag{252}$$

for all $L, M \in \mathfrak{so}(V)$, $s \in S$, $v, w \in V$ and $A, B \in \mathfrak{r}$. Here $X : V \to \mathfrak{so}(V)$ is a linear map which geometrically corresponds to the choice of a basis of T_pM consisting of (the values at the point of) some Killing vectors, R is the Riemann curvature tensor and the maps β, γ and ρ are as determined in Theorems 10 and 11. If $H^- = 0$ then $\mathfrak{r}$ is not included in $\mathfrak{a}$ and the Lie brackets in (252) involving elements $A, B \in \mathfrak{r}$ need to be disregarded. We also note that $\rho = 0$ in this case. It is clear from (252) that the Killing superalgebra (resp. the extended Killing superalgebra) is isomorphic to a filtered subdeformation $\mathfrak{g}$ of $\mathfrak{p}$ (resp. $\hat{\mathfrak{p}}$), where the components of positive degree of the Lie brackets are given by the underbraced elements and the underlying graded Lie superalgebra is $\mathfrak{a}$ with its natural Lie brackets.

Now, one can easily check (compare e.g. with Lemma 3 of [39]) that the space of cocycles $Z^{d,2}(\mathfrak{a}_-, \mathfrak{a}) = 0$ for all even $d \geqslant 4$ and that $\mathfrak{a}$ is a full prolongation of degree 2 in the sense of [40], that is $H^{d,1}(\mathfrak{a}_-, \mathfrak{a}) = 0$ for all $d \geqslant 2$ (see also Lemma 7). It follows that the infinitesimal filtered deformation is completely determined by (the orbit under $\mathrm{Spin}(V) \times \mathrm{Sp}(1)$ of) the corresponding cohomology class in $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$. Associated to the natural inclusion $\imath$ of $\mathfrak{a}$ into $\mathfrak{p}$ or $\hat{\mathfrak{p}}$ there is a map in cohomology

$$\imath_* : H^{2,2}(\mathfrak{a}_-, \mathfrak{a}) \to \Lambda^3 V \oplus \left(V \otimes \odot^2 \Delta\right)\,, \tag{253}$$

which is easily seen to be injective (remember that $\mathfrak{a}_- = \mathfrak{p}_- = \hat{\mathfrak{p}}_- = V \oplus S$ by maximal supersymmetry). Hence, the infinitesimal filtered deformation is completely determined by (the orbit of) $\varphi|_p \in V \otimes \odot^2\Delta$ in the first branch and $H|_p \in \Lambda^3 V$ in the second branch. With some more effort, we may show that this data do actually allow to recover the whole filtered deformation, up to isomorphisms of filtered subdeformations of $\mathfrak{p}$ or $\hat{\mathfrak{p}}$. The claim relies on $Z^{4,2}(\mathfrak{a}_-, \mathfrak{a}) = 0$, the fact that $\mathfrak{a}$ is a full prolongation of degree 2 and the general results of [40] – we refer to [19, Proposition 10] for more details.

We have seen that the maximally supersymmetric backgrounds (M, g, H, φ) of Theorem 27 have different associated Killing superalgebras, up to isomorphisms of filtered subdeformations of $\mathfrak{p}$ if $H^- = 0$ and $\hat{\mathfrak{p}}$ if

$\mathsf{H}^- \neq 0$. It is certainly possible to give an explicit description of these Lie superalgebras using (252), in the spirit of [19, §5] and [25, §4], but we don't do that for the sake of brevity.

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